BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VI Numéro 7

VARSOVIE 1958

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique.

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

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PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE – WARSZAWA
MIODOWA 10

Nakład 887+113 egz.	*Rękopis dostarczono 6. VI. 1958*
Ark. wyd. 4, ark. druk. 3⅞	*Podpisano do druku 4. VIII. 1958*
Papier bezdrzewny sat. 80 g kl. III	*Druk ukończono w sierpniu 1958*
Format B5, 70×100 cm	*Zam. prod. 403/58 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
FOLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHEMATICS

On the Incidence Axioms of Various Geometries

by

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Presented by A. MOSTOWSKI on January 7, 1958

In the present paper an outline is given of a certain theory G with axioms depending on numerical parameters: m, p, q. By substituting appropriate numbers for the values of these parameters we obtain systems of axioms, models of which are various known geometries, e. g.

m	p	q	M o d e l
$n+1$	1	0	affine n-dimensional geometry
$n+1$	0	0	projective n-dimensional geometry
$n+2$	2	0	Möbius n-dimensional geometry *)
$n+2$	1	2	Laguerre n-dimensional geometry *)
$n+3$	2	2	Lie n-dimensional geometry *)

The theory outlined here is not a categorical one (not even in the case of fixed values for m, p, q) and its axioms do not constitute, for given m, p, q, a complete system of axioms of a corresponding geometry. In the particular models the notions characterized by axioms of the theory G are analogous to the incidence notions of the affine geometry. For example, assuming for the parameters the following values:

$$m = 4, \quad p = 1, \quad q = 0$$

we may prove, on the basis of the theory G, all the incidence axioms of the three-dimensional affine geometry ([4], p. 3).

The following problems are — among others — open to discussion:

PROBLEM 1. *To give, for particular models of theory G, axioms, which in conjunction with those of the theory G form complete (categorical) systems of axioms of the corresponding geometries.*

*) All these geometries are defined in [2].

PROBLEM 2. *Is it possible to add to the axioms of the theory G listed below some further axioms common to all models given in this theory. Would the axioms completed in such a way constitute, after each separate determination of the value of the parameters m, p, q, a categorical theory, and would they present a complete system of axioms of an appropriate model?*

The theory G completed in the above mentioned manner could be considered as a universal geometry of some kind.

Although the theory G as outlined below, does not comply with the requirements of Problem 2, it covers a sufficiently large fragment of universal geometry and makes it possible to demonstrate several general theorems valid for all geometries being models of that theory.

The preliminaries thus being fixed, we give in what follows the axioms of the theory G.

Let us consider two sets, X and Y. For the sake of convenience we adopt the symbol x with some distinctive indices for denoting elements of the set X, and the symbol y for denoting elements of the set Y. A third primitive notion is that of a binary relation whose domain is X and counterdomain Y. We shall denote this relation by the symbol $\sim$ and we propose to call it the *incidence relation*. Thus, if $x \sim y$, we shall say that x is incident to y, or that both x and y are mutually incident *). In order to simplify the notations we shall write $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_l)$ instead of writing separately: $x_i \sim y_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. If the condition $x \sim y$ is not fulfilled, we write $x \nsim y$.

Definition 1. We call the elements $(x_1, x_2, \ldots, x_k)$ independent and we write $I(x_1, \ldots, x_k)$ if for each $i = 1, 2, \ldots, k$ there is a y_i, such that

(1) $$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \sim y_i$$

and

(2) $$x_i \nsim y_i .$$

We define in the same manner the independence of the elements $y_1, \ldots, y_k$.

We may admit that the empty set 0 fulfills the condition of independency, i. e. that $I(0)$.

Definition 2. The formula $(x_1, \ldots, x_k) s (y_1, \ldots, y_l)$ means that the following conditions are satisfied:

(1) $I(x_1, \ldots, x_k)$,
(2) $I(y_1, \ldots, y_l)$,
(3) $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_l)$,
(4) $x_i \nsim x_j$ and $y_i \nsim y_j$ for each different value of i and j.

*) It might be of some advantage, in order to facilitate our exposition, to consider X as the set of all points in the three dimensional space, Y — as the set of planes, and the relation $\sim$ as the incidence relation of points and planet.

Before formulating the axioms, let us recall that the theory depends on three parameters m, p, q. For their values we may assume any integers satisfying the conditions:

$$p \geqslant 0, \quad q \geqslant 0, \quad p+q < m .$$

AXIOM 1. *There are* $x_1, \ldots, x_p, y_1, \ldots, y_q$ *fulfilling the condition*

$$(x_1, \ldots, x_p) s (y_1, \ldots, y_q) .$$

AXIOM 2. *If* $(x_1, \ldots, x_k) s (y_1, \ldots, y_l)$, $(x_0) \sim (y_1, \ldots, y_l)$ *and* $(x_1, \ldots, x_k) \sim (y_0)$, *where* $k \geqslant 0$, $b \geqslant 0$, $k+1 = m$, *then*

$$x_0 \sim y_0 .$$

AXIOM 3. *If* $(x_1, \ldots, x_k) s (y_1, \ldots, y_l)$, *where* $k \geqslant p$, $1 \geqslant q$, $k+1 < m$, *then there exist elements* x_{k+1} *and* y_{l+1} *such that* $(x_1, \ldots, x_k, x_{k+1}) s (y_1, \ldots, y_l)$ *and* $(x_1, \ldots, x_k) s (y_1, \ldots, y_l, y_{l+1})$.

Let us remark that the axioms given above determine in full only the „incidence part" of particular geometries.

On the basis of these axioms it is possible to work out a certain lattice with properties similar to those of a metric lattice ([1] p. 76, [3] p. 68, 69). For $p = q = 0$, that is for the case of projective geometry, this lattice is strictly metrical.

In order to obtain this lattice it is necessary to introduce some auxiliary notions.

Definition 3. We call simplex any set of m elements

$$S_{k,l} = (x_1, \ldots, x_k, y_1, \ldots, y_l)$$

such that

$$k \geqslant p, \; 1 \geqslant q, \; k+1 = m, \; (x_1, \ldots, x_k) s (y_1, \ldots, y_l) .$$

To denote simplexes we use symbols of the form $S^i_{k,l}$, where k and l denote numbers of elements x_μ, y_ν belonging to the simplex considered, and i denotes the different simplexes.

Definition 4. Two simplexes

$$S^1_{k_1,l_1} = (x^1_1, \ldots, x^1_{k_1}, y^1_1, \ldots, y^1_{l_1}) \quad \text{and} \quad S^2_{k_2,l_2} = (x^2_1, \ldots, x^2_{k_2}, y^2_1, \ldots, y^2_{l_2}) ,$$

are called equivalent if

$$(x^1_1, \ldots, x^1_{k_1}, x^2_1, \ldots, x^2_{k_2}) \sim (y^1_1, \ldots, y^1_{l_1}, y^2_1, \ldots, y^2_{l_2}) .$$

Being of the equivalence type, this relation divides the set of simplexes into disjoint classes of abstraction such that $S^1_{k_1,l_1}$ and $S^2_{k_2,l_2}$ belong to the same class if, and only if, they are equivalent.

Definition 5. $\hat{S}_{k,l}$ denotes the class of all simplexes equivalent to the simplex $S_{k,l}$.

It is possible to demonstrate for these abstraction classes the following

THEOREM 1. *If the simplexes*

$$S^1_{k_1,l_1} = (x^1_1, \ldots, x^1_{k_1}, y^1_1, \ldots, y^1_{l_1})$$

and

$$S^{1'}_{k'_1,l'_1} = (x^{1'}_1, \ldots, x^{1'}_{k'_1}, y^{1'}_1, \ldots, y^{1'}_{l'_1})$$

and also the simplexes

$$S^2_{k_2,l_2} = (x^2_1, \ldots, x^2_{k_2}, y^2_1, \ldots, y^2_{l_2})$$

and

$$S^{2'}_{k'_2,l'_2} = (x^{2'}_1, \ldots, x^{2'}_{k'_2}, y^{2'}_1, \ldots, y^{2'}_{l'_2})$$

are equivalent, then the condition

$$(x^1_1, \ldots, x^1_{k_1}) \sim (y^2_1, \ldots, y^2_{k_2})$$

implies the condition

$$(x^{1'}_1, \ldots, x^{1'}_{k'_1}) \sim (y^{2'}_1, \ldots, y^{2'}_{k'_2}).$$

Definition 6. We say that $\hat{S}^1_{k_1,l_1}$ precedes $\hat{S}^2_{k_2,l_2}$ if the elements of of the simplexes:

$$S^1_{k_1,l_1} = (x^1_1, \ldots, x^1_{k_1}, y^1_1, \ldots, y^1_{l_1})$$

and

$$S^2_{k_2,l_2} = (x^2_1, \ldots, x^2_{k_2}, y^2_1, \ldots, y^2_{l_2})$$

fulfill the condition

$$(x^1_1, \ldots, x^1_{k_1}) \sim (y^2_1, \ldots, y^2_{l_2}).$$

We denote this relation by $\hat{S}_{k_1,l_1} \prec \hat{S}^2_{k_2,l_2}$.

It is easy to verify that the last relation partially orders the set of abstraction classes $\hat{S}_{k,l}$.

It is further possible to demonstrate — using but few auxiliary propositions — that the set of abstraction classes constitutes a lattice, provided it possesses the elements 0 and 1 fulfilling for each element $\hat{S}_{k,l}$ the condition

(*) $$0 \prec \hat{S}_{k,l} \prec 1.$$

The element 0 exists for $p = 0$. It is namely the abstraction class $\hat{S}_{0,m}$. The abstraction class $\hat{S}_{m,0}$ for $q = 0$ is the element 1. In other cases it is necessary to add to the abstraction classes additional elements 0, 1 — in order to obtain the lattice. Thus, we may formulate the following

THEOREM 2. *If we add to the set of abstraction classes the elements* 0 *and* 1, *fulfilling for each* $\hat{S}_{k,l}$ *the condition* (*), *such a set constitutes a lattice.*

The interpretation of the lattice, as defined above, varies with various values of the parameters. Thus, e. g. for an n-dimensional projective geometry and for an n-dimensional affine geometry, the elements of the lattice constitute multidimensional planes; the relation ($\prec$) means that one plane is contained in the other. This is a generally known geometrical interpretation of the lattice ([1] p. 110, 116, [3] p. 8, 25, Ex. 7, 8). In the Möbius geometry, spheres and planes with various dimensions constitute the elements of the lattice.

The lattice as defined above possesses some interesting properties common, of course, to all its models. The most important one is the following.

Let $\mathrm{mod}(\hat{S}_{k,l})$ (the module of $\hat{S}_{k,l}$) be defined as k.

THEOREM 3. *If*

$$\mathrm{mod}(\hat{S}_{k_1,l_1} \frown \hat{S}^2_{k_2,l_2}) \geqslant p \qquad \textit{and} \qquad m - \mathrm{mod}(\hat{S}^1_{k_1,l_1} \smile \hat{S}^2_{k_2,l_2}) \geqslant q\,,$$

then

$$\mathrm{mod}(\hat{S}^1_{k_1,l_1} \frown \hat{S}^2_{k_2,l_2}) + \mathrm{mod}(\hat{S}^1_{k_1,l_1} \smile \hat{S}^2_{k_2,l_2}) = \mathrm{mod}(\hat{S}^1_{k_1,l_1}) + \mathrm{mod}(\hat{S}^2_{k_2,l_2})\,.$$

(The symbols $\frown$ and $\smile$ mean here a joint and a meet of the elements of the lattice in the ordinary sense ([1] p. 16, [3] p. 20, 24).

The interpretation of the notion of the module in various models of the theory G is strictly bound to the number of dimensions of the considered element of the lattice. Thus, for example, in both the affine and the projective geometries the dimension of a plane $S_{k,l}$ is equal to its module less one.

The development of the theory G makes it possible to ascertain the number of connections between the particular geometries and to demonstrate theorems common to different geometries. Moreover, by substituting new values for the parameters m, p, q we are able to obtain certain new geometries with properties analogous to those of the geometries already known. The investigation of different interpretations and applications of such geometries might prove both useful and of marked interest.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHEMATICS

On E-Compact Spaces

by

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Presented by K. KURATOWSKI on April 26, 1958

In 1948 Hewitt [1] introduced the notion of Q-space; this notion has found many fruitful applications in the investigation of spaces of real-valued continuous functions. Further development of the theory of Q-spaces has shown that there are many analogies between Q-spaces and compact spaces; many theorems are valid both for compact spaces and Q-spaces. We note some of them:

a) *A closed subset of a compact space (Q-space) is a compact space (Q-space)* *).

b) *The product of arbitrary many compact spaces (Q-spaces) is a compact space (Q-space)* **).

c) *For each space X there is an extension βX (νX), which contains X as a dense subset; it is a compact space (Q-space) and has the property of each bounded continuous real-valued function (arbitrary continuous real-valued function) defined on X admitting a continuous extension over βX (νX).*

In this paper we present a common generalization of the theory of compact spaces and Q-spaces. This generalization gives, in particular, a uniform method of proving the above theorems.

1. Definitions and notations

All topological spaces under consideration are supposed to be Hausdorff.

Y^X denotes the set of all continuous functions defined on X and with values from Y; $[Y^X]$ denotes the union of all $(Y^n)^X$ $(n = 1, 2, \ldots)$. Notice that, if X is a subset of X_1 and each function $f \in Y^X$ admits an extension $f^* \in Y^{X_1}$, then each function $h \in [Y^X]$ admits an extension $h^* \in [Y^{X_1}]$.

*) This statement for Q-spaces is due to Katětov [2] and Shirota [3].

**) This statement for Q-spaces is due to Shirota [3].

Let E be a topological space. We say that a space X is *completely regular with respect to* E (or, shortly, *E-completely regular*) provided that for each closed set $A \subset X$ and each point $p \in X \setminus A$ there exists a function $h \in [E^X]$ such that $h(p) \notin \overline{h(A)}$. We denote by $\mathfrak{C}(X)$ the class of all E-completely regular spaces.

Notice that the property "to be an E-completely regular space" is hereditary and "productive" (i. e. if X, X_t $(t \in T)$ are E-completely regular spaces and $A \subset X$, then A and the product $\prod_t X_t$ are also E-completely regular spaces). Indeed, it is shown in [5] that *a space X is E-completely regular if, and only if, X can be embedded into some Cartesian potency E^m.*

Note. If in the definition of the E-complete regularity the condition "$h \in [E^X]$" is replaced by the condition "$h \in E^X$", then we obtain the definition of the *strong E-complete regularity*. The authors have no knowledge of whether the strong E-complete regularity is equivalent to E-complete regularity. It may be easily verified that the answer is positive if, and only if, the following holds true:

For each space E, E^2 is strongly E-completely regular.

I denotes the closed unit interval $[0, 1]$; R — the real line; D — the two-point discrete space; N denotes the space of all non-negative integers (the discrete space of the power $\aleph_0$).

2. E-compact spaces

Let E be a topological space. A space X is said to be *E-compact* provided that $X \in \mathfrak{C}(E)$ and that there exists no space $Y \in \mathfrak{C}(E)$ which contains X as a dense proper subset and has the property such that each function $f \in E^X$ admits an extension $f^* \in E^Y$.

This definition is similar to that of Q-spaces given by Katětov in [2]. In the sequel it will be shown (Theorem 2) that a space X is E-compact if, and only if, X is homeomorphic to a closed subset of some Cartesian potency of the space E. In particular, it implies that a closed subset of an E-compact space and the product of arbitrary many E-compact spaces are again E-compact (Theorems 1 and 3). Theorem 4 asserts the existence of an extension of an E-completely regular space X which is similar to βX in the case of compact spaces, and to νX in the case of Q-spaces.

Examples:

(i) *A space X is I-compact if, and only if, X is compact.*

(ii) *A space X is R-compact if, and only if, X is a Q-space.*

(iii) *A space X is D-compact if, and only if, W is 0-dimensional and compact.*

(iv) *A space X is N-compact if, and only if, X is 0-dimensional and a Q-space.*

Proof of (iv) is based on the following lemmas:

LEMMA 1. *If X is a 0-dimensional space* *) *and $p_0 \in \beta X \setminus X$, then the space $X \cup \{p_0\}$ is also 0-dimensional.*

Proof. Since X is 0-dimensional, X can be embedded into the Cartesian potency $D^{\mathfrak{m}}$ (see [4] or [5], Theorem 4). Denote as Y the closure of X in $D^{\mathfrak{m}}$. Then Y is a continuous image of X, $Y = h(\beta X)$, where $h(p) = p$ for each p in X. Let $q_0 = h(p_0) \in Y \setminus X$ (this follows from Lemma 3). Hence, the space $X \cup \{q_0\}$ is a continuous one-to-one image of $X \cup \{p_0\}$. Since $X \cup \{q_0\}$ is 0-dimensional, $X \cup \{p_0\}$ is also 0-dimensional.

LEMMA 2. *If X is a 0-dimensional Q-space, then for each point $p_0 \in \beta X \setminus X$ there is a continuous function f on βX which takes only values $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and such that $f(p_0) = 0$, $f(p) \neq 0$ for each p in X.*

Proof. Since X is a Q-space, there is a function $g \in I^{\beta X}$ with $g(p_0) = 0$ and $g(p) \neq 0$ for each p in X (see [6], Proposition). Let $F_n = \left\{p \in X: |g(p)| \geqslant \frac{1}{n}\right\}$. Since $X \cup \{p_0\}$ is 0-dimensional and $p_0 \bar{\in} F_n$, one can found an open-closed set F_n^* with $F_n \subset F_n^*$ and $p_0 \bar{\in} F_n^*$. Of course, it may be supposed that $F_1^* \subset F_2^* \subset \ldots$ Let f_n be defined by equalities: $f_n(p) = \frac{1}{n}$ for $p \in F_n$ and $f_n(p) = 0$ for $p \in X \setminus F_n$ and let $f(p) = \sup_n f_n(p)$. Then $\bar{f}$ is continuous on X. Denoting as f the continuous extension of $\bar{f}$ over βX, we obtain the required function.

Proof of (iv). Suppose X is N-compact. Since $X \in \mathfrak{C}(N)$, X is 0-dimensional. Let Y be any space which contains X as a dense proper subset. Since X is N-compact, there exists a function $f \in N^X$ which admits no extension $f^* \subset N^Y$. Of course, $f \in R^X$. Suppose that f admits an extension $f^{**} \in R^Y$. Since X is dense in Y, $\overline{f^{**}(Y)} = \overline{f^{**}(X)} = \overline{f(X)} \subset N$ (the bar indicates the closure with respect to R), hence $f^{**} \in N^Y$ and this leads to a contradiction. Thus, X is a Q-space.

Conversely, suppose X is a 0-dimensional Q-space. Then $X \in \mathfrak{C}(N)$. Let $Y \in \mathfrak{C}(N)$ be any space which contains X as a dense proper subset. Then Y is a continuous image of some subset Z of βX, $Y = h(Z)$, where $h(p) = p$ for each p in X. Let $p_0 \in Y \setminus X$ and $g_0 = h^{-1}(p_0)$. Then $g_0 \in Z \setminus X$, hence there exists $f \in R^{\beta X}$ such that $f(g_0) = 0$, $f(p) \neq 0$ for $p \in X$, and f takes only values $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$ Let f_1 be defined on X by the equality $f_1(p) = \frac{1}{f(p)}$. Then $f_1 \in N^X$ but f_1 admits no extension $f^* \in N^Y$.

3. Some properties of E-compact spaces

THEOREM 1. *If X is an E-compact space and F is a closed subset of X, then F is also an E-compact space.*

*) A space is said to be 0-*dimensional* provided that it possesses a basis of closed-open neighbourhoods.

Proof. Suppose F is not an E-compact space. Then there exists a space $F^* \in \mathfrak{C}(E)$ which contains F as a dense proper subset and such that each function $f \in E^F$ admits an extension $f^* \in E^{F^*}$. Let p_0 be any point of $F^* \setminus F$. Of course, one can assume that p_0 is distinct from each point of X. For each f in $[E^X]$, let $\bar{f}$ = the continuous extension of $f|F$ over F^*. We define the closure operation in $X^* = X \cup \{p_0\}$:

$p_0 \in \bar{M}$ if $\bar{f}(p_0) \in \overline{\bar{f}(M)}$ for each f in $[E^X]$;

$p \in \bar{M}$ if $p \neq p_1$ and p belongs to the closure of $M \cap X$ with respect to X.

It can be easily verified that such an operation satisfies all the axioms of the closure operation. Indeed, the conditions $\bar{M} \cup \bar{N} \subset \overline{M \cup N}$, $M \subset \bar{M}$, $\bar{O} = 0$ are obvious; on the other hand, from $f(\overline{M \cup N}) = \overline{f(M)} \cup \overline{f(N)}$ there follows $\overline{M \cup N} \subset \bar{M} \cup \bar{N}$. Moreover, if $p_0 \bar{\in} \bar{M}$, then there is $f \in [E^X]$ with $\bar{f}(p_0) \in \overline{\bar{f}(M)}$. By the continuity of f, $\bar{f}(\bar{M}) \subset \overline{\bar{f}(M)}$, hence $\overline{\bar{f}(M)} \subset \overline{f(M)}$, and therefore $p_0 \bar{\in} \bar{\bar{M}}$. It shows that $\bar{\bar{M}} \in \bar{M}$.

It is plain that X^* contains X as a subspace (i. e. the topology of X^* relativized to X agrees with the original topology of X).

We shall show that X is a T_1-space. Of course, $\overline{\{p_0\}} = \{p_0\}$. In order to prove that $\overline{\{p\}} = \{p\}$ for each p in X, it suffices to show that there is a function $f \in [E^X]$ such that $\bar{f}(p_0) \neq \bar{f}(p)$. If $p \in X \setminus F$, then there is f in $[E^X]$ with $\bar{f}(p) \bar{\in} \overline{\bar{f}(F)}$. But $\bar{f}(p_0) \in \overline{\bar{f}(F)}$, and it follows that $\bar{f}(p) \neq \bar{f}(p_0)$. If $p \in F$, then, in virtue of $F^* \in C(E)$, there is a function $g \in [E^{F^*}]$ with $g(p) \neq g(p_0)$. Let U be a neighbourhood of $g(p)$ such that $g(p_0) \bar{\in} \bar{U}$. By the continuity of g, the set $g^{-1}(\bar{U})$ is closed in F^*, it does not contain p and it contains p_0 in its interior. The set $g^{-1}(\bar{U}) \cap X$ is closed in X, hence there is $f \in [E^X]$ such that $f(p) \bar{\in} f(\overline{g^{-1}(\bar{U}) \cap X})$. Since

$p_0 \in g^{-1}(\bar{U}) \cap X$ (the closure with respect to F^*),

$\bar{f}(p_0) \in \bar{f}(g^{-1}(\bar{U}) \cap X)$, and it follows $\bar{f}(p_0) \neq \bar{f}(p)$.

The space X has the property such that each function $f \in E^X$ admits an extension $f^* \in E^{X^*}$; namely, it is enough to put $f^*(p_0) = \bar{f}(p_0)$. The continuity of f^* immediately follows from the definition of the closure operation in X^*.

It can be easily shown that X is an E-completely regular space. In fact, suppose that $p \in \bar{M}$. If $p \neq p_0$, then there is f in $[E^X]$ with $f(p) \in \overline{f(M \cap X)}$, hence $f(p) \in \overline{f(\bar{M})}$, and if $p = p_0$, then there is f in $[E^X]$ with $\bar{f}(p) \in \overline{\bar{f}(\bar{M})}$, hence $f^*(p) \in \overline{f^*(\bar{M})}$; f^* denotes the continuous extension of f over X^*.

Of course, X^* contains X as a dense subset and this leads to a contradiction. Thus, F is a Q-space.

STATEMENT 1. *Suppose $X \in \mathfrak{C}(E)$ and $\mathfrak{m}$ is the power of the set $[E^X]$. Then X can be embedded into the Cartesian potency $E^{\mathfrak{m}}$ in such a way that each function $f \in E^X$ admits an extension $f^* \in E^{\bar{X}}$ (the bar indicates the closure with respect to $E^{\mathfrak{m}}$).*

Proof. Each member f of $[E^X]$ takes values from E^n (n is an integer); we denote this space as E_f. Then the product $\boldsymbol{P}_f E_f$ is homeomorphic to $E^{\mathfrak{m}}$. We define the embedding mapping φ as the mapping which carries a point $p \in X$ into the point $x \in E^{\mathfrak{m}} = \boldsymbol{P}_f E_f$ whose f-th co-ordinate is equal to $f(p)$. Since X is an E-completely regular, φ is a homeomorphism. Since each function in E^X can be considered as a co-ordinate-function on $\varphi(X)$, it can be continuously extended over $\overline{\varphi(X)}$.

LEMMA 3. *If S is a dense proper subset of a space P, then there exists no mapping $h \in S^P$ such that $h(p) = p$ for each p in S.*

Proof. Let p_0 be any point of $P \setminus S$. Then $q_0 = h(p_0) \neq p_0$. There is a neighbourhood U of p_0 with $q_0 \bar{\in} \overline{U}$. Since S is dense in P, $p_0 \in \overline{S \cup U}$, hence $q_0 \in h(\overline{S \cap U}) \subset h(\overline{S \cap U}) = \overline{S \cap U} \subset \overline{U}$ and this leads to a contradiction.

STATEMENT 2. *The Cartesian potency $E^{\mathfrak{m}}$ is E-compact.*

Proof. Suppose that there exists a space $E^* \in \mathfrak{C}(E)$ which contains $E^{\mathfrak{m}}$ as a dense proper subset and has the property of each function $f \in E^{(E^{\mathfrak{m}})}$ admitting an extension $f^* \in E^{E^*}$. Then each co-ordinate-function $p_\xi \in E^{(E^{\mathfrak{m}})}$ admits an extension $p_\xi^* \in E^{E^*}$. Let h be the mapping of E^* into $E^{\mathfrak{m}}$ which carries a point $p \in E^*$ into the point $x \in E^{\mathfrak{m}}$ whose ξ-th co-ordinate is equal to $p_\xi^*(p)$. Of course, $h(p) = p$ for each p in $E^{\mathfrak{m}}$, and this contradicts Lemma 3.

THEOREM 2. *A space X is E-compact if, and only if, X is homeomorphic to a closed subset of $E^{\mathfrak{m}}$ ($\mathfrak{m}$ is the power of the set $[E^X]$).*

Proof. If X is homeomorphic to a closed subset of $E^{\mathfrak{m}}$, then, by Statement 2 and Theorem 1, X is E-compact. Conversely, suppose X is E-compact. By Statement 1, X can be embedded into $E^{\mathfrak{m}}$ in such a way that each function $f \in E^X$ admits an extension $f^* \in E^X$. It follows that X coincides with $\bar{X}$.

THEOREM 3. *If $X_\xi (\xi \in \Xi)$ is a family of E-compact spaces, then the product $\boldsymbol{P}_\xi X_\xi$ is also E-compact.*

Proof. By Theorem 2 each space X_ξ is homeomorphic to a closed subset of the Cartesian potency $E_\xi = E^{\mathfrak{m}_\xi}$, hence the product $\boldsymbol{P}_\xi X_\xi$ is homeomorphic to a closed subset of the product $P_\xi E_\xi$. But the product $\boldsymbol{P}_\xi E_\xi$ is again a Cartesian potency of E, and, by Theorem 2, $P_\xi X_\xi$ is E-compact.

THEOREM 4. *Suppose $X \in \mathfrak{C}(E)$. Then*

a) *there exists an extension $\nu_E X \in \mathfrak{C}(E)$ of X which satisfies the following conditions:*

(i) *$\nu_E X$ is an E-compact space and it contains X as a dense subset;*

(ii) *each function $f \in E^X$ admits an extension $f^* \in E^{\nu_E X}$,*

(b) *if Y is any E-compact space, then each function $f \in Y^X$ admits an extension $f^* \in Y^{\nu_E X}$,*

c) *the extension $\nu_E X$ is uniquely determined by the conditions* (i) *and* (ii) *in the sense that, if Z is any extension of X satisfying the conditions* (i) *and* (ii), *there exists a homeomorphism h of $\nu_E X$ onto Z such that $h(p) = p$ for each p in X.*

Proof. Part a) immediately follows from Statements 1 and 2 and Theorem 1.

Part b) Since Y is an E-compact space, one can assume that Y is a closed subset of $E^{\mathfrak{m}}$. Let f be any function in Y^X. Denote as f_ξ the function which assigns to a point $p \in X$ the ξ-th co-ordinate of the point $f(p)$. Of course, $f_\xi \in E^X$, hence f_ξ admits an extension $f^* \in E^{\nu_E X}$. Denote as f^* the function which assigns to a point $p \in \nu_E X$ the point $x \in E^{\mathfrak{m}}$ whose ξ-th co-ordinate is equal to $f^*_\xi(p)$. Of course, $f^* \in (E^{\mathfrak{m}})^{\nu_E X}$ and it is an extension of f. Since Y is closed in $E^{\mathfrak{m}}$ and X is dense in $\nu_E X$, $f^* \in ({}_E X) \subset Y$, thus $f^* \in Y^{(\nu_E X)}$ and b) follows.

Part c) Suppose Z is any extension of X satisfying (i) and (ii). The identity mapping of X onto itself admits extensions $h \in Z^{(\nu_E X)}$ and $h^* \in (\nu_E X)^z$. Of course, $h^{-1} = h^*$ on X, and since X is dense in $\nu_E X$, h is a homeomorphism of $\nu_E X$ onto Z which leaves all points of X fixed.

4. Applications

From Example (iv) and Theorem 2 we obtain the following theorem (compare with the result of A. H. Stone [8]):

The Cartesian potency $N^{\mathfrak{c}}$ is non-normal.

Proof. A completely regular space X which is a non-normal 0-dimensional Q-space having an enumerable dense subset was described in [7]. By (iv), X is N-compact and, of course, the family $[N^X]$ is of the power $\mathfrak{c}$. By Theorem 2, X can be embedded into $N^{\mathfrak{c}}$ as a closed subset and it follows that $N^{\mathfrak{c}}$ is non-normal.

5. Final remarks

Notice that if E is a completely regular space, then each space $X \in \mathfrak{C}(E)$ is also completely regular. In this case, the definition of E-compact spaces can be formulated as follows:

If E is a completely regular space, then space $X \in \mathfrak{C}(E)$ is E-compact if, and only if, for each p in $\beta X \setminus X$ there exists a function $f \in (\beta E)^{\beta X}$ such that $f(X) \subset E$ and $f(p) \in \beta E \setminus E$.

Proof. Suppose X is an E-compact space, let p be any point of $\beta X \setminus X$ and $Y = X \cup \{p\}$. There is a function $f \in E^X$ which admits no extension $f \, E^Y$. But f admits an extension $f_1 \in (\beta E)^{\beta X}$ and it follows that $f_1(p) \in \beta E \setminus E$.

Conversely, let Y be any space which contains X as a dense proper subset. Then there exists a continuous function h defined on a set Z, $X \subset Z \subset \beta X$, such that $h(Z) = Y$ and $h(p) = p$ for $p \in X$. Let p be a point of Z with $h(p) \in Y \setminus X$. By the assumption, there is a function $g \in (\beta E)^{\beta X}$ with $g(X) \subset E$, $g(p) \bar{\in} E$. Of course, the function $f = g|X \in E^X$ admits no extension $f^* \in E^Y$.

From Theorem 3 follows a corollary concerning the effectiveness of a certain topological theorem. It is known that the proof of the product theorem for compact spaces is ineffective (i. e. it essentially depends upon the axiom of choice). On the other hand, the proof of Theorem 3 is quite effective. Substituting in this theorem E by I, we obtain the product theorem for compact spaces; of course, provided the definition of compact spaces, given in Example (i), is adopted. But the equivalence of this definition with the ordinary one effectively follows from the existence of βX for each completely regular X. Thus we obtain the following

Corollary. *The existence of βX for each completely regular X effectively implies the product theorem for compact spaces.*

Added in proof. The negative solution of the problem: *is always E^2 strongly completely regular with respect to E* can be given with the use of the following result which was kindly communicated to me by Professor J. de Groot during his stay in Poland:

There exists a subset E of the Euclidean plane which contains more than one point and has the property that each continuous mapping of E into itself is either the identity or a constant mapping.

Now, let us take this set E and let x_1 and x_2 be distinct points of E. Let us take the point $p_0 = \langle x_1, x_2 \rangle$ in E^2 and the set $A = \{\langle x_1, x_1 \rangle, \langle x_2, x_2 \rangle\} \subset E^2$. Of course, A is a closed subset of E^2 and $p_0 \bar{\in} A$. Suppose there exists a continuous mapping f of E^2 into E such that $f(p_0) \bar{\in} \overline{f(A)}$. Then the mapping f restricted to the set $\{\langle x, x_2 \rangle : x \in E\}$ is non-constant; therefore, by the fundamental property of E, it is the identity, i. e. $f(x, x_2) = x$; in particular, $f(p_0) = f(x_1, x_2) = x_1$. Similarly, f restricted to the set $\{\langle x_1, x \rangle : x \in E\}$ is non-constant, therefore $f(x_1, x) = x$ and, in particular, $f(p_0) = f(x_1, x_2) = x_2$ and it leads to a contradiction. Thus E^2 is not strongly completely regular with respect to E.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHEMATICS

A Central Limit Theorem for some Stochastic Processes

by

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Presented by H. STEINHAUS on May 5, 1958

1. Let $\{Y_k(t),\ 0 \leqslant t \leqslant 1\}$ $(k = 1, 2, ...)$ be a sequence of real, separable ([5] p. 51), independent and equally distributed stochastic processes. The last property means that, for $s = 1, 2, 3, ...$ and arbitrary points $t_1, ..., t_s$ from the interval $[0, 1]$ the random vectors $\{Y_k(t_1), ..., Y_k(t_s)\}$ $(k = 1, 2, ...)$ are equally distributed. Denote:

$$F(t) = EY_k(t), \qquad (0 \leqslant t \leqslant 1)$$

$$X_n(t) = \frac{1}{n}\sum_{k=1}^{n} Y_k(t)\,,$$

$$\xi_n(t) = \sqrt{n}[X_n(t) - F(t)]\,.$$

We shall denote by $D[0, 1]$ the complete, separable metric space of real functions defined on the interval $[0, 1]$, having both right- and left-hand limits at each point t and continuous from the left (at $t = 0$ from the right) with the metric d introduced by Prohorov ([11] p. 228).

THEOREM 1. *Let* $\{Y_k(t),\ 0 \leqslant t \leqslant 1\}$ $(k = 1, 2, ...)$ *be a sequence of real, separable, independent and equally distributed stochastic processes. We assume that:*

α) *the following relations are true for arbitrary* t_1, t_2, *where* $0 < t_1 \leqslant t_2 < 1$,

$$EY_k(t_1)\,Y_k(t_2) - F(t_1)F(t_2) = u(t_1)v(t_2)\,, \tag{1}$$

where $u(t), v(t)$ *is a continuous and monotonically increasing function.*

$$E[Y_k(t_2) - Y_k(t_1)]^2 \leqslant C_1(t_2 - t_1)\,, \tag{2}$$

where C_1 *does not depend on* t.

β) *For arbitrary* $t_1, t_2, t_3 \in [0, 1]$, *where* $t_1 < t_2 < t_3$, *the relation*

$$E\,|Y_k(t_2) - Y_k(t_1)|^j\,|Y(t_3) - Y_k(t_2)|^m \leqslant C_2(t_2 - t_1)^{j/2}(t_3 - t_2)^{m/2} \tag{3}$$

holds for $j, m = 1, 2$, *and* C_2 *does not depend on* t.

γ) *The probability that $Y_k(t)$ is continuous from the left at those points t at which the left-hand limit exists (at the point $t = 0$ from the right if $Y_k(+0)$ exists) is equal to* 1.

Then, with probability 1 *the realizations of $\xi_n(t)$ belong to $D[0, 1]$ and*

$$P^{\xi_n} \Rightarrow P^{\xi_0}, \tag{4}$$

where $\{\xi_0(t),\ 0 \leqslant t \leqslant 1\}$ is a real, separable, Gaussian stochastic process with

$$E\xi_0(t) = 0 \qquad (0 \leqslant t \leqslant 1) \tag{5}$$

$$E\xi_0(t_1)\xi_0(t_2) = u(t_1)v(t_2) \qquad (0 \leqslant t_1 \leqslant t_2 \leqslant 1)$$

and where P^{ξ_n} and P^{ξ_0} are probability measures generated in space $D[0, 1]$ by the finite dimensional distributions of $\xi_n(t)$ and $\xi_0(t)$, respectively ([7] § III, 4).

Proof. In virtue of the central limit theorem the relation

$$\lim_{n \to \infty} P^{\xi_n}(\xi_n(t_1) < z_1, \ldots, \xi_n(t_s) < z_s) = P^{\xi_0}(\xi_0(t_1) < z_1, \ldots, \xi_0(t_s) < z_s) \tag{6}$$

holds for $s = 1, 2, 3, \ldots$, arbitrary $t_1, t_2, \ldots, t_s \in [0, 1]$ and real $z_1, \ldots, z_s$. Consider, for arbitrary $t_1, t_2, t_3 \in [0, 1]$, where $t_1 < t_2 < t_3$, the expression

$$\begin{aligned}
&A(t_1, t_2, t_3) = E[\xi_n(t_2) - \xi_n(t_1)]^2[\xi_n(t_3) - \xi_n(t_2)]^2 \\
&= \frac{1}{n}\{E[Y_k(t_2) - Y_k(t_1)]^2[Y_k(t_3) - Y_k(t_2)]^2 + 3[F(t_2) - F(t_1)]^2[F(t_3) - F(t_2)]^2 \\
&+ [F(t_2) - F(t_1)]^2 D^2[Y_k(t_3) - Y_k(t_2)] + [F(t_3) - F(t_2)]^2 D^2[Y_k(t_2) - Y_k(t_1)] \\
&- 2[F(t_2) - F(t_1)]E[Y_k(t_2) - Y_k(t_1)][Y_k(t_3) - Y_k(t_2)]^2 \\
&- 2[F(t_3) - F(t_2)]E[Y_k(t_2) - Y_k(t_1)]^2[Y_k(t_3) - Y_k(t_2)] \\
&+ 4[F(t_2) - F(t_1)][F(t_3) - F(t_2)]E[Y_k(t_2) - Y_k(t_1)][Y_k(t_3) - Y_k(t_2)]\} \\
&+ 2\left(1 - \frac{1}{n}\right)\{\tfrac{1}{2} D^2[Y_k(t_2) - Y_k(t_1)] D^2[Y_k(t_3) - Y_k(t_2)] \\
&+ E[[Y_k(t_2) - Y_k(t_1)][Y_k(t_3) - Y_k(t_2)] - [F(t_2) - F(t_1)][F(t_3) - F(t_2)]]^2\}.
\end{aligned}$$

Relations (2) and (3) imply

$$|A(t_1, t_2, t_3)| \leqslant C(t_2 - t_1)(t_3 - t_2) < C(t_3 - t_1)^2, \tag{7}$$

where $C = 8C_1^2 + 8C_1C_2 + 4C_2[C_1 + 3C_2$. According to a theorem of Centsov ([1], Theorem 2), in virtue of the continuity (with probability 1) of the realizations of $\xi_0(t)$ ([4] p. 398), relations (6) and (7) imply the relation

$$\lim_{n \to \infty} P^{\xi_n}(g(t) \leqslant \xi_n(t) \leqslant f(t);\ 0 \leqslant t \leqslant 1) = P^{\xi_0}(g(t) \leqslant \xi_0(t) \leqslant f(t);\ 0 \leqslant t \leqslant 1), \tag{8}$$

where the functions $g(t)$ and $f(t)$ are defined for $s = 1, 2, 3, \ldots$, arbitrary divisions of the interval $[0, 1]$ by points $0 = \tau_0 < \tau_1 < \ldots < \tau_s = 1$, and for

arbitrary real vectors $(b_1, \ldots, b_s)$, $(c_1, \ldots, c_s)$ with $b_m < c_m$ $(m = 1, \ldots, s)$ by the equalities

$$(9) \qquad \left.\begin{aligned} g(t) &= b_m \\ f(t) &= c_m \end{aligned}\right\} \; m = 1, \ldots, s; \qquad \tau_{m-1} < t \leqslant \tau_m .$$

Now, relation (7) and assumption γ) imply in virtue of a theorem of Ĉentsov ([1] Theorem 1) that, with probability 1, the realizations of $\xi_n(t)$ belong to space $D[0, 1]$.

Denote by S the class of sets of functions $\varphi(t)$ belonging to $D[0, 1]$ which satisfy the inequality

$$(10) \qquad g(t) < \varphi(t) < f(t) ,$$

where $g(t)$ and $f(t)$ are arbitrary functions defined by (9). We shall show that S forms a basis in space $D[0, 1]$. This fact has merely been stated by Kolmogorov and Prohorov ([9] p. 124).

Since I have not found the proof thereof in the literature, it is given below.

Let us consider the metric d in $D[0, 1]$ introduced by Prohorov [11]. By definition, for $\varphi_1, \varphi_2 \in D[0, 1]$

$$(11) \qquad d(\varphi_1, \varphi_2) = \varrho(\Gamma_{\varphi_1}, \Gamma_{\varphi_2}) + L(F_{\varphi_1}, F_{\varphi_2}) ,$$

where Γ_{φ_i} $(i = 1, 2)$ is the graph of φ_i and

$$\varrho(\Gamma_{\varphi_1}, \Gamma_{\varphi_2}) = \max\Big[\sup_{M_1 \in \Gamma_{\varphi_1}} \inf_{M_2 \in \Gamma_{\varphi_2}} |M_1 - M_2|, \sup_{M_2 \in \Gamma_{\varphi_2}} \inf_{M_1 \in \Gamma_{\varphi_1}} |M_1 - M_2|\Big] ,$$

whereas $L(F_{\varphi_1}, F_{\varphi_2})$ is the Lévy-distance [10] of the non-decreasing functions $F_{\varphi_i}(z)$ which are defined as follows: denote

$$\widetilde{\omega}_{\varphi_i}(\delta) = \sup_{\{\Delta : |\Delta| \leqslant \delta\}} \sup_{t \in \Delta} \min\big[|\varphi_i(t) - \varphi_i(t_1)|, |\varphi_i(t_2) - \varphi_i(t)|\big] ,$$

where $\Delta = [t_1, t_2]$ and the supremum is extended over all intervals Δ with $|\Delta| \leqslant \delta$. Then, for $i = 1, 2$,

$$F_{\varphi_i}(z) = \begin{cases} \widetilde{\omega}_{\varphi_i}(e^z) & (z \leqslant 0) \\ \widetilde{\omega}_{\varphi_i}(1) & (z > 0) . \end{cases}$$

Let now φ_0 be some point in $D[0, 1]$, and let $\varepsilon > 0$ be some given number. Take s so large and vectors $(b_1, \ldots, b_s)$, $(c_1, \ldots, c_s)$ such that

$$(12) \qquad \max_m (c_m - b_m) < \varepsilon$$

and that $\varphi_0(t)$ satisfies (10) for $g(t)$ and $f(t)$ defined by (9) with the vectors considered.

Denote now by A the set of elements of $D[0, 1]$ satisfying (10) with $g(t)$ and $f(t)$ just considered. Clearly, $\varphi_0 \in A$. We have further, in virtue of (12)

$$(13) \qquad \begin{aligned} &\varrho(\Gamma_g, \Gamma_f) < \varepsilon , \\ &\sup_z |F_g(z) - F_f(z)| < \varepsilon , \end{aligned}$$

and consequently

$$L(F_g, F_f) < \varepsilon . \tag{14}$$

Relations (13) and (14) imply $d(g, f) < 2\varepsilon$, and consequently $\delta(A) < 2\varepsilon$, where $\delta(A)$ is the diameter of the set A. Since ε is arbitrary it has been proved that S forms a basis in $D[0, 1]$.

It is evident that, from $A_1 \in S$ and $A_2 \in S$, the relation $(A_1 \cap A_2) \in S$ follows. Using a theorem of Prohorov ([11], Theorem 1.9) we obtain relation (4) as a consequence of relation (8). Thus Theorem 1 has been proved.

The special case of Theorem 1 with

$$Y_k(t) = \begin{cases} 1 & (t > T_k) \\ 0 & (t \leqslant T_k), \end{cases} \tag{15}$$

where $\{T_k\}$ is a sequence of independent random variables uniformly distributed in the interval $[0, 1]$ has been proved by Donsker [3].

As a conclusion from Theorem 1 we obtain

THEOREM 2. *Let* $\{Y_k(t),\ 0 \leqslant t \leqslant 1\}$ $(k = 1, 2, 3, \ldots)$ *be a sequence of real, separable, independent and equally distributed stochastic processes, and let the assumption* α) *with* $u(t) = t$, $v(t) = 1 - t$, *as well as assumption* β) *of Theorem* 1 *be satisfied. Then, for each* $\lambda > 0$,

$$\lim_{n\to\infty} P^{\xi_n}(\sup_t |\xi_n(t)| < \lambda) = Q(\lambda) = \sum_{r=-\infty}^{\infty} (-1)^r \exp(-2\lambda^2 r^2) . \tag{16}$$

Proof. We have obtained relation (8) from the assumptions of this theorem. Since, as Kolmogorov [8] and Doob [4] have shown,

$$P^{\xi_0}(\max_t |\xi_0(t)| < \lambda) = Q(\lambda) , \tag{17}$$

relation (16) is obtained.

Theorem 2 generalizes a well-known theorem of Kolmogorov [8] concerning functions $Y_k(t)$ defined by (15).

Note that the assumption that the $Y_k(t)$ are equally distributed can be omitted from Theorems 1 and 2. However, in this case, formulae (1)-(3) should be replaced by more complicated expressions which will ensure the correctness of (6) and (7).

2. We consider j $(j \geqslant 2)$ independent sequences of stochastic processes $\{Y_{pk}(t),\ 0 \leqslant t \leqslant 1\}$ $(p = 1, 2, \ldots, j;\ k = 1, 2, 3, \ldots)$ studied in the preceding section. Denote

$$N = (n_1, \ldots, n_j),$$

$$\xi_{pn}(t) = \sqrt{n}\left[\frac{1}{n}\sum_{k=1}^{n} Y_{pk}(t) - F(t)\right], \tag{18}$$

$$\eta_{Ni}(t) = \sum_{p=1}^{j} \beta_{Nip}\, \xi_{pn_p}(t) \qquad (i = 1, \ldots, r)\,, \tag{19}$$

where β_{Nip} are real constants. Let us write $N \to \infty$ instead of $n_1 \to \infty, \ldots, n_j \to \infty$. Denote further by K the Cartesian product of j spaces $D[0,1]$ and take as distance between two points $(\varphi_1, \ldots, \varphi_j)$, $(\varphi'_j, \ldots, \varphi'_j)$ of K the expression

$$\sqrt{\sum_{p=1}^{j} d^2(\varphi_p, \varphi'_p)}\,,$$

where d is Prohorov's distance. Let K' denote the space of vector-functions $(\psi_{N1}(t), \ldots, \psi_{Nr}(t))$, where $\psi_{Ni}(t)$ $(i = 1, \ldots, r)$ is defined on K by linear relation (19). Finally, denote by $P^{\eta_{N1}, \ldots, \eta_{Nr}}$ the probability measure in K' generated by the vector process $(\eta_{N1}(t), \ldots, \eta_{Nr}(t))$.

THEOREM 3. *Let* $\{Y_{pk}(t),\ 0 \leqslant t \leqslant 1\}$ $(p = 1, 2, \ldots, j;\ k = 1, 2, 3, \ldots)$ *be* j *independent sequences of stochastic processes satisfying all assumptions of Theorem* 1. *If the limits*

$$\lim_{N \to \infty} \beta_{Nip} = \beta_{ip} \qquad (p = 1, 2, \ldots, j;\ i = 1, \ldots, r) \tag{21}$$

exist, then:

(i) *As* $N \to \infty$ *the relation*

$$P^{\eta_{N1}, \ldots, \eta_{Nr}} \Longrightarrow P^{\eta_{01}, \ldots, \eta_{0r}} \tag{22}$$

holds, where $P^{\eta_{01}, \ldots, \eta_{0r}}$ *is the probability measure in* K' *generated by the vector process* $(\eta_{01}(t), \ldots, \eta_{0r}(t))$ *with*

$$\eta_{0i}(t) = \sum_{p=1}^{j} \beta_{ip}\, \xi_{0p}(t)\,, \tag{23}$$

and where $\xi_{0p}(t)$ $(p = 1, 2, \ldots, j)$ *are independent, real, separable Gaussian stochastic processes with means and covariance-matrix given by* (5).

(ii) *If, moreover, we have in formula* (1) $u(t) = t$, $v(t) = 1 - t$, *and the* β_{ip}*-s satisfy the equalities*

$$\sum_{p=1}^{j} \beta_{hp} \beta_{ip} = \delta_{hi}\,, \qquad \textit{Kronecker delta } (h, i = 1, \ldots r) \tag{24}$$

where, of course, $r \leqslant j$, *then, for arbitrary positive* $\lambda_1, \ldots, \lambda_r$, λ *the relations*

$$\lim_{N \to \infty} P^{\eta_{N1}, \ldots, \eta_{Nr}}(\sup_t |\eta_{Ni}(t)| < \lambda_i;\ i = 1, \ldots, r) = \prod_{i=1}^{r} Q(\lambda_i)\,, \tag{25}$$

$$\lim_{N \to \infty} P^{\eta_{N1}, \ldots, \eta_{Nr}}(\max_{1 \leqslant i \leqslant r} \sup_t |\eta_{Ni}(t)| < \lambda) = [Q(\lambda)]^r, \tag{26}$$

hold, where $Q(\lambda)$ *is given by* (16).

Proof. Denote by $P^{\xi_{1n_1},\ldots,\xi_{jn_j}}$ and $P^{\xi_{01},\ldots,\xi_{0j}}$, respectively, the probability measures in K generated by the vector processes $(\xi_{1n_1}(t), \ldots, \xi_{jn_j}(t))$ and $(\xi_{01}(t), \ldots, \xi_{0j}(t))$. From the independence of the components of these vectors and from (4), which holds for the sequences $P^{\xi_{pn_p}}$ $(p = 1, \ldots, j)$ in virtue of the assumptions of Theorem 3, we obtain as $N \to \infty$

$$P^{\xi_{1n_1},\ldots,\xi_{jn_j}} = P^{\xi_{1n_1}} \times \ldots \times P^{\xi_{jn_j}} \Rightarrow P^{\xi_{01}} \times \ldots \times P^{\xi_{0j}} = P^{\xi_{01},\ldots,\xi_{0j}}. \tag{27}$$

Let π_N denote the transformation of space K into space K' given by (19). In virtue of (21) the sequence $\{\pi_N(\varphi_1, \ldots, \varphi_j)\}$, where $(\varphi_1, \ldots, \varphi_j) \in K$, converges uniformly on every compact set of K to the transformation $\pi(\varphi_1, \ldots, \varphi_j)$ obtained by replacing in (19) the β_{Nip} by the limits β_{ip}. Since

$$P^{\eta_{N1},\ldots,\eta_{Nr}} = P^{(\xi_{1n_1},\ldots,\xi_{jn_j})\pi_N},$$

$$P^{\eta_{01},\ldots,\eta_{0r}} = P^{(\xi_{01},\ldots,\xi_{0j})\pi}$$

we obtain relation (22) from a theorem of Prohorov ([11], Theorem 1.10) and from (27). Assertion (i) of Theorem 3 is thus proved.

In order to prove assertion (ii) let us note that (24) and (23) imply that the processes $\eta_{0i}(t)$ $(i = 1, \ldots, r)$ are independent, normal and satisfy (5). Consequently,

$$P^{\eta_{01},\ldots,\eta_{0r}} = P^{\eta_{01}} \times \ldots \times P^{\eta_{0r}}. \tag{28}$$

Since the realizations of $\eta_{0i}(t)$ are, with probability 1, continuous, the transformation

$$(\psi_{01}, \ldots \psi_{0r}) \to (\sup_t |\psi_{01}|, \ldots, \sup_t |\psi_{0r}|),$$

which transforms space K' into the r-dimensional Euclidean space is almost everywhere $(P^{\eta_{01},\ldots,\eta_{0r}})$ continuous. It follows from a theorem of Prohorov ([11] Theorem 1.8) that the expression on the left of (25) is equal to

$$P^{\eta_{01},\ldots,\eta_{0r}}(\max_t |\eta_{0i}(t)| < \lambda_i; \quad i = 1, \ldots, r).$$

Taking into account (17) and (28) we obtain (25). Relation (26) follows immediately from (25) by taking $\lambda_1 = \ldots = \lambda_r = \lambda$.

The special case of Theorem 3 with $Y_{pk}(t)$ given by (15) with $r = j-1$ and $\sum_{p=1}^{j} \beta_{Nip} \sqrt{n_p} = 0$ $(i = 1, \ldots, j-1)$ has been given in [6] and [2] and, for the particular case $j = 2$, in [12].

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHÉMATIQUE

Sur une caractérisation algébrique des espaces métriques

par

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Presenté par K. BORSUK le 14 Mai 1958

Une certaine nouvelle caractérisation algébrique des espaces métriques de puissance au plus $2^{\aleph_0}$ peut être obtenue sur la base des propriétés des métriques aléatoires introduites dans [3]. En ce qui concerne les notations nous renvoyons le lecteur à [2] .

Soit X un ensemble quelconque, $X^2 = X \times X$, F l'ensemble de toutes les transformations de X^2 dans l'espace R de tous les nombres réels et $\boldsymbol{F}$ la plus petite σ-algèbre de sous-ensembles de F engendrée par la classe

$$\{\{f : f \in F,\ f(x, y) < r\} : x, y \in X,\ r \in R\}.$$

Si $X \neq 0$ et φ est une mesure de probabilité dans $\boldsymbol{F}$, l'espace de probabilité $(F, \boldsymbol{F}, \varphi)$ est appelé fonction aléatoire sur X^2. On dit qu'elle peut être réalisée dans l'ensemble $M \subset F$ de toutes les métriques dans X, s'il existe une mesure de probabilité μ_φ dans la σ-algèbre $\boldsymbol{M} = M \cap \boldsymbol{F}$ de sorte que $\mu_\varphi(M \cap E) = \varphi(E)$ pour tout $E \in \boldsymbol{F}$. Cette définition est équivalente à $\bar{\varphi}(M) = 1$, $\bar{\varphi}$ désignant la mesure extérieure engendrée par φ, ou, d'après [3], équivalente aux conditions

$$\varphi\{f : f \in F,\ f(x, x) = 0\} = 1 \quad \text{pour} \quad x \in X,$$

$$\varphi\{f : f \in F,\ f(x, y) > 0\} = 1 \quad \text{pour} \quad x, y \in X,\ x \neq y,$$

$$\varphi\{f : f \in F,\ f(x, y) + f(z, y) \geqslant f(x, z)\} = 1 \quad \text{pour} \quad x, y, z \in X$$

qui doivent être satisfaites simultanément.

Si $1 \leqslant p(X) \leqslant 2^{\aleph_0}$, $p(X)$ désignant la puissance de X, l'ensemble $S \subset M$ de toutes les métriques séparables dans X est non-vide. On dit que la fonction aléatoire $(F, \boldsymbol{F}, \varphi)$ peut être réalisée dans S, s'il existe une mesure de probabilité σ_φ dans la σ-algèbre $\boldsymbol{S} = S \cap \boldsymbol{F}$ de sorte que $\sigma_\varphi(S \cap E) = \varphi(E)$ pour $E \in \boldsymbol{F}$.

D'après [3], *si* $1 \leqslant p(X) \leqslant 2^{\aleph_0}$, *les réalisabilités de* $(F, \boldsymbol{F}, \varphi)$ *dans* M *et* S *sont équivalentes.*

Définissons $\xi(f, x, y) = f(x, y)$ pour $f \in F$ et $x, y \in X$. La fonction $\xi(\cdot, x, y)$ est une variable aléatoire par rapport à $\boldsymbol{F}$ et les réductions $\xi^M(\cdot, x, y)$ ou $\xi^S(\cdot, x, y)$ de $\xi(\cdot, x, y)$ sur M ou S sont des variables aléatoires par rapport à $\boldsymbol{M}$ et $\boldsymbol{S}$ respectivement pour $x, y \in X$.

Si la fonction aléatoire $(F, \boldsymbol{F}, \varphi)$ peut être réalisée dans S, on a

$$\int_F \xi(f, x, y) d\varphi = \int_M \xi^M(f, x, y) d\mu_\varphi = \int_S \xi^S(f, x, y) d\sigma_\varphi$$

pour $x, y \in X$ et si, de plus,

$$\int_F \xi(f, x, y) d\varphi < \infty \quad \text{pour} \quad x, y \in X,$$

ces intégrales définissent une métrique dans X.

Soit $1 \leqslant p(X) \leqslant 2^{\aleph_0}$, soit $\varrho \in M$ une métrique fixée *non-séparable* dans X et soit φ la mesure de probabilité dans $\boldsymbol{F}$ qui est définie de façon que $\varphi(E) = 1$, si $\varrho \in E \in \boldsymbol{F}$, et $\varphi(E) = 0$, si $E \in \boldsymbol{F}$, $\varrho \in E' = F - E$.

Il est intéressant de constater que même dans ce cas la fonction aléatoire $(F, \boldsymbol{F}, \varphi)$ peut être réalisée dans l'ensemble S de toutes les métriques séparables dans X et nous avons

$$\int_S \xi(f, x, y) d\sigma_\varphi = \varrho(x, y)$$

pour $x, y \in X$.

Supposons que $1 \leqslant p(X) \leqslant 2^{\aleph_0}$ et considérons les réalisations $(S, \boldsymbol{S}, \sigma)$ des fonctions aléatoires dans X, où σ est une mesure élémentaire de probabilité qui ne prend que les valeurs 1 ou 0. Chaque mesure élémentaire de probabilité σ dans $\boldsymbol{S}$ engendre univoquement le σ-idéal maximal $\boldsymbol{I}_\sigma = \{E : E \in \boldsymbol{S},\ \sigma(E) = 0\}$ dans $\boldsymbol{S}$ et son dual $\boldsymbol{I}'_\sigma = \{E : E' \in \boldsymbol{I}_\sigma\} = \{E : E \in \boldsymbol{S},\ \sigma(E) = 1\}$ et inversement tout σ-idéal maximal $\boldsymbol{I}$ dans $\boldsymbol{S}$ détermine univoquement la mesure élémentaire de probabilité $\sigma_{\boldsymbol{I}}$ dans $\boldsymbol{S}$ de sorte que $\sigma_{\boldsymbol{I}}(E) = 0$ si $E \in \boldsymbol{I}$ et $\sigma_{\boldsymbol{I}}(E) = 1$ si $E' \in \boldsymbol{I}$.

Le résultat principal de la presente note est le suivant:

Théorème. *Si $1 \leqslant p(X) \leqslant 2^{\aleph_0}$, il existe une transformation biunivoque t de l'ensemble M de toutes les métriques dans X sur la classe Γ de tous les σ-idéaux maximaux dans $\boldsymbol{S}$ de façon que*

$$\int_S \xi^S(f, x, y) d\sigma_{t(\varrho)} = \varrho(x, y)$$

pour $x, y \in X$ et $\varrho \in M$, $\sigma_{t(\varrho)}$ désignant la mesure élémentaire de probabilité engendrée par le σ-idéal maximal $t(\varrho) \in \Gamma$.

Ce théorème contient une caractérisation globale des espaces métriques de puissance au plus $2^{\aleph_0}$, qui est analogue à la définition des nombres réels sur la base des coupures de Dedekind dans l'espace des nombres rationnels. Chaque métrique est essentiellement un σ-idéal ma-

ximal dans une σ-algèbre appropriée de sous-ensembles de l'ensemble des métriques séparables et inversement. Une caractérisation semblable peut être obtenue facilement en appliquant le théorème cité et la représentation de Banach-Mazur [1] des espaces métriques séparables dans l'espace universel C des fonctions réelles continues dans l'intervalle-unité fermée. Chaque métrique est essentiellement un σ-idéal dans une σ-algèbre appropriée de sous-ensembles de l'ensemble de toutes les transformations biunivoques de X sur les sous-ensembles de C et inversement. Donc, C est universel par rapport à la classe de tous les espaces métriques de puissance au plus $2^{\aleph_0}$.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHEMATICS

Theory of Models with Infinitary Operations and Relations

by

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Presented by A. MOSTOWSKI on April 6, 1958

1. Let β be an ordinal number and let $\Delta = \langle \alpha_0, \alpha_1, \ldots, \alpha_\sigma, \ldots \rangle_{\sigma<\beta}$ be a sequence of the type β the elements of which are arbitrary ordinal numbers. By the *rank* of Δ we shall denote the least initial number ϱ such that $\bar{\alpha}_\sigma \leqslant \bar{\varrho}$, $\sigma < \beta$. By the *dimension* of Δ we shall denote the least initial number γ which, for all $\sigma < \beta$ and $\alpha \leqslant \alpha_\sigma$, satisfies the following conditions: (i) $\alpha_\sigma < \gamma$, and (ii) γ is not cofinal with any ordinal number $\alpha \leqslant \alpha_\sigma$.

The cardinal number m is called *Δ-regular*, if (1) for all $\sigma < \beta$ and for an arbitrary set Γ with $\overline{\overline{\Gamma}} \leqslant \bar{\alpha}_\sigma$, m_σ are cardinal numbers such that $m_\sigma < m$ for $\sigma \in \Gamma$. Then $\sum_{\sigma \in \Gamma} m_\sigma < m$.

Let ϱ and γ be the rank and dimension of Δ. It is easy to see that:

$$\gamma = \begin{cases} \varrho, \text{ if } \bar{\varrho} \text{ is } \Delta\text{-regular} \\ \omega_{\tau+1}, \text{ if } \varrho = \omega_\tau \text{ and } \bar{\varrho} \text{ is not } \Delta\text{-regular} \end{cases} \tag{1.1}$$

$$\bar{\gamma} \text{ is } \Delta\text{-regular (and also regular).} \tag{1.2}$$

2. A sequence $\boldsymbol{A} = \langle A, F_0, F_1, \ldots F_\sigma, \ldots \rangle_{\sigma<\beta}$ in which A is a non-empty set and F_σ is an operation of the type α_σ in A, is called an *algebra of the type* $\Delta = \langle \alpha_\sigma, \alpha_1, \ldots, \alpha_\sigma, \ldots \rangle_{\sigma<\beta}$. Let β, ϱ and γ be order, rank and dimension of $\Delta \cdot \beta$, ϱ and γ are called order, rank and dimension of every algebra of the type Δ.

Let $\boldsymbol{A} = \langle A, F_0, F_1, \ldots F_\sigma, \ldots \rangle_{\sigma<\beta}$ be an algebra of the type Δ. By $F_\sigma(X)$, for $X \subset A$, we denote the set of all elements in A which are values of operation F_σ for elements in X. Let A_0 be a non-empty subset of A.

(2.1) We define by induction the sequence of sets $\langle A_0, A_1, \ldots A_\xi, \ldots \rangle_{\xi<\gamma}$ such that

$$A_\xi = \sum_{\tau<\xi} A_\tau + \sum_{\eta<\beta} F_\eta\Big(\sum_{\tau<\xi} A_\tau\Big).$$

The sets A_ξ, $\xi < \gamma$, are called *the Borel-classes of set A_0 in algebra $\boldsymbol{A}$*, in particular A_ξ is called ξ-Borel-class of A_0 in $\boldsymbol{A}$.

By $\{A_0\}$ we denote the subalgebra of $\boldsymbol{A}$ which is generated by set A_0. The following theorem can be proved

$$\{A_0\} = \Big\langle \sum_{\xi<\gamma} A_\xi, F_0, F_1, \dots F_\sigma, \dots \Big\rangle_{\sigma<\beta}. \tag{2.2}$$

From (2.2) and (1.1) follow the theorems:

$$\overline{\overline{\{A_0\}}} \leqslant 2^{\overline{\overline{A}}_0 \cdot \bar\beta \cdot \bar\varrho} \tag{2.3}$$

(2.4) *If $\overline{\overline{A}}_0 \cdot \beta \cdot \bar\varrho$ is Δ-regular, then $\overline{\overline{\{A_0\}}} \leqslant \overline{\overline{A}}_0 \cdot \bar\beta \cdot \bar\varrho$.*

The sign $\leqslant$ in (2.3) and (2.4) cannot be replaced by the sign $<$.

3. Let $\mathfrak{A}$ be the class of all algebras of the type $\Delta = \langle \alpha_0, \alpha_1, \dots, \alpha_\sigma \dots \rangle_{\sigma<\beta}$ of order β, rank ϱ and dimension γ.

(3.1) *An algebra $\boldsymbol{W} = \langle W, F_0, F_1, \dots, F_\sigma, \dots \rangle_{\sigma<\beta}$ is called an absolutely free algebra of the type Δ, if*

(1) *$\boldsymbol{W}$ is an algebra of the type Δ,*

(2) *there exists a set $N_0 \subset W$ such that $\boldsymbol{W}$ is generated by N_0 and the elements in N_0 are not the values of operations F_σ, $\sigma < \beta$, for elements in W;*

(3) *for all $\eta, \lambda < \beta$ and all $w_0, w_1, \dots w_\xi, \dots \xi < \alpha_\eta, w'_0, w'_1, \dots w'_\xi, \dots \xi < \alpha_\lambda$ in W, if $F_\eta(w_0, w_1, \dots w_\xi, \dots)_{\xi<\alpha_\eta} = F_\lambda(w'_0, w'_1, \dots, w'_\xi, \dots)_{\xi<\alpha_\lambda}$,*

$$\text{then} \quad F_\eta = F_\lambda \quad \text{and} \quad w_\xi = w'_\xi \quad \text{for} \quad \xi < \alpha_\eta = \alpha_\lambda.$$

The set N_0 is called *free set of generators of algebra $\boldsymbol{W}$* and $\boldsymbol{W}$ is also called an *absolutely free algebra of the type Δ freely generated by set N_0*. The following theorems can be proved:

(3.2) *There exist absolutely free algebras of the type Δ freely generated by the sets of arbitrary power.*

(3.3) *If $\boldsymbol{W}$ is an absolutely free algebra of the type Δ freely generated by N_0, then every mapping $\varphi_0(N_0) \subset \boldsymbol{A} \in \mathfrak{A}$ can be extended to a homomorphism of $\boldsymbol{W}$ in $\boldsymbol{A}$.*

(3.4) *Two absolutely free algebras of the type Δ freely generated by the sets of the same power are isomorphic,*

(3.5) *Every subalgebra $\boldsymbol{W}'$ of an absolutely free algebra of the type Δ is also an absolutely free algebra of the type Δ freely generated by the elements in $\boldsymbol{W}'$ which are not values of operations for elements in $\boldsymbol{W}'$.*

Let $\boldsymbol{W}$ be an absolutely free algebra of the type Δ freely generated by a set N. The elements in $\boldsymbol{W}$ are called *the terms*, the elements in N are *variables*. Let ϑ be a term, i. e. $\vartheta \in \boldsymbol{W}$. The least subset $N_0 \subset N$ such

that $\vartheta \in \{N_0\}$ is called the *support of* ϑ and is denoted by $s(\vartheta)$. We prove by induction that

$$\overline{\overline{s(\vartheta)}} < \overline{\gamma} \quad \text{(and also } \overline{\overline{s(\vartheta)}} \leqslant \overline{\varrho}\text{)}. \tag{3.6}$$

4. Let β, β_1 be arbitrary ordinal numbers and let A be a non-empty set. A sequence $\boldsymbol{M} = \langle A, F_0, F_1, \ldots F_\sigma, \ldots; R_0, R_1, \ldots R_\xi, \ldots\rangle_{\sigma<\beta,\, \xi<\beta_1}$ in which F_σ is an operation of the type α_σ in A and R_ξ is a relation of the type λ_ξ in A, is called *a model of the type* $\Delta_1 = \langle \alpha_0, \alpha_1, \ldots \alpha_\sigma \ldots; \lambda_0, \lambda_1, \ldots \lambda_\xi \ldots\rangle_{\sigma<\beta,\, \xi<\beta_1}$. α_σ and λ_ξ are arbitrary ordinal numbers; we assume only $\lambda_0 = 2$. Let ϱ_1 and γ_1 be the rank and dimension of Δ_1. ϱ_1 and γ_1 are called *rank and dimension of every model of the type* Δ_1. The models $\boldsymbol{M}$ in which R_0 is the relation of identity are called *models with identity*.

Let $\mathfrak{K}_1$ be the class of all models of the type Δ_1. Let η, τ be arbitrary ordinal numbers. For the class $\mathfrak{K}_1$ may be constructed a logic $P_{\eta,\tau}$ within which models belonging to $\mathfrak{K}_1$ can be discussed. We construct this logic as follows. Let N be a set of power $\aleph_\tau$. The elements of the algebra $\boldsymbol{W} = \langle W, f_0, f_1, \ldots f_\sigma, \ldots\rangle_{\sigma<\beta}$ absolutely free of the type $\Delta = \langle \alpha_0, \alpha_1, \ldots \alpha_\sigma \ldots\rangle_{\sigma<\beta}$ freely generated by set N are called the terms. We consider the elements of N as variables. Let $r_0^*, r_1^*, \ldots, r_\xi^*, \ldots\ \xi < \beta_1$ be a sequence of the type β_1 composed by arbitrary different elements. By r_ξ we denote the set of all pairs $\langle r_\xi^*, (\vartheta_\sigma)_{\sigma<\lambda_\xi}\rangle$, where $(\vartheta_\sigma)_{\sigma<\lambda_\xi}$ is a sequence of the type λ_ξ the elements of which are terms. The elements of $r_\xi, \xi < \beta_1$ are considered as *atomic formulae*. We shall denote the atomic formula $\langle r_\xi^*, (\vartheta_\sigma)_{\sigma<\lambda_\xi}\rangle$ by $r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}$. $U_\tau = \sum_{\xi<\beta_1} r_\xi$ is the set of all atomic formulae. It depends only on τ and we call it the set of *atomic formulae corresponding to* τ. The atomic formulae $r_0(\vartheta_0, \vartheta_1)$ are called *equations*.

The formulae of logic $P_{\eta,\tau}$ *are the elements of the algebra*

$$\boldsymbol{P}_{\eta,\tau} = \langle P_{\eta,\tau}, \to, ', \underset{2}{\vee}, \underset{2}{\wedge}, \ldots \underset{\xi}{\vee}, \underset{\xi}{\wedge}, \ldots;\ \xi < \omega_\eta\rangle$$

absolutely free of the type $\langle 2, 1, 2, 2, \ldots \xi, \xi, \ldots;\ \xi < \omega_\eta\rangle$ *freely generated* by U_τ.

The operations $\to, ', \underset{\xi}{\vee}, \underset{\xi}{\wedge}$ are called respectively *implication, negation, disjunction* and *conjunction of the type* ξ.

The support of an atomic formula $a = r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}$ is the set $s(a) = \sum_{\sigma<\lambda_\xi} s(\vartheta_\sigma)$, where $s(\vartheta_\sigma)$ is the support of the term ϑ_σ.

Let $U(p)$, for $p \in P_{\eta,\tau}$, be the least subset $U \subset U_\tau$ such that $p \in \{U\}$, where $\{U\}$ denotes the subalgebra of $P_{\eta,\tau}$ generated by U. *The support of formula* $p \in P_{\eta,\tau}$ is $s(p) = \sum_{a \in U(p)} s(a)$. The elements in $s(p)$ are variables appearing in formula p.

The following theorems can be proved:

(4.1) $$\overline{\overline{s(p)}} \leqslant \bar{\varrho}_1 \cdot \aleph_\eta .$$

(4.2) *If* $\aleph_\eta \geqslant \bar{\gamma}_1$ *and* $\aleph_\eta$ *is regular, then* $\overline{\overline{s(p)}} < \aleph_\eta$.

Let φ be a mapping in W^N. By (3.3) φ can be extended to a homomorphism $h_\varphi(\boldsymbol{W}) \subset \boldsymbol{W}$ which is called φ — *substitution in* $\boldsymbol{W}$. The term $h_\varphi(\vartheta) = \vartheta_{(\varphi)}$, for $\vartheta \in W$, is called φ-*substitution of* ϑ.

Let φ_1 be a mapping of U_τ in $P_{\eta,\tau}$ such that

$$\varphi_1\left(r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}\right) = r_\xi(\vartheta_{0(\varphi)}, \vartheta_{1(\varphi)}, \ldots \vartheta_{\sigma(\varphi)}, \ldots)_{\sigma<\lambda_\xi} .$$

φ_1 may be extended to a homomorphism $h(\boldsymbol{P}_{\eta,\tau}) \subset \boldsymbol{P}_{\eta,\tau}$ which is called φ-*substitution in* $\boldsymbol{P}_{\eta,\tau}$. The formula $h(p) = p_{(\varphi)}$ for $p \in P_{\eta,\tau}$, is called φ-*substitution of* p.

Now we shall give a definition of *validity of formulae of logic* $P_{\eta,\tau}$.

Let ψ be a mapping of N in a model

$$\boldsymbol{M} = \langle A, F_0, F_1, \ldots F_\sigma, \ldots;\ R_0, R_1, \ldots R_\xi, \ldots\rangle_{\sigma<\beta,\,\xi<\beta_1}$$

belonging to class $\mathfrak{K}_1$. ψ is also a mapping of N in the algebra $\boldsymbol{A} = \langle A, F_0, F_1, \ldots F_\sigma, \ldots\rangle_{\sigma<\beta}$ of model $\boldsymbol{M}$ and therefore by (3.3) ψ can be extended to a homomorphism $h_\psi(\boldsymbol{W}) \subset \boldsymbol{A}$.

Let

$$\boldsymbol{B} = \langle (0,1), \Rightarrow, \Delta, \underset{2}{\cup}, \underset{2}{\cap}, \ldots \underset{\xi}{\cup}, \underset{\xi}{\cap} \ldots;\ \xi < \omega_\eta\rangle$$

be a two-element Boolean algebra similar to $\boldsymbol{P}_{\eta,\tau}$ (i. e. $\aleph_\eta$-complete for all $\xi < \eta$).

Let ψ_1 be a mapping of U_τ in $\boldsymbol{B}$ such that

$$\psi_1\left(r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}\right) = \begin{cases} 1, & \text{if } R_\xi(h_\psi(\vartheta_0), h_\psi(\vartheta)_1, \ldots h_\psi(\vartheta_\sigma)\ldots)_{\sigma<\lambda_\xi} \text{ holds} \\ 0, & \text{in the opposite case} \end{cases}$$

ψ_1 can be extended to a homomorphism g of the algebra $\boldsymbol{P}_{\eta,\tau}$ in $\boldsymbol{B}$. We denote by $g_p^{(\boldsymbol{M})}(\psi)$ the value of g for $p \in P_{\eta,\tau}$. $g_p^{(\boldsymbol{M})}(\psi)$ is a function of $\psi \in A^N$ which is called *value-function of formula* p *in model* $\boldsymbol{M}$. A formula p is ψ-*valid in* $\boldsymbol{M}$, if $g_p^{(\boldsymbol{M})}(\psi) = 1$. A formula p *is valid in* $\boldsymbol{M}$, if for all $\psi \in A^N$ $g_p^{(\boldsymbol{M})}(\psi) = 1$. A set $X \subset P_{\eta,\tau}$ *is valid in* $\boldsymbol{M}$, if every formula in X is valid in $\boldsymbol{M}$. $\boldsymbol{M} \in \mathfrak{K}_1$ is called *a model of a set* $X \subset P_{\eta,\tau}$, if X is valid in $\boldsymbol{M}$. By $\mathfrak{K}_0(X)$, where $\mathfrak{K}_0 \subseteq \mathfrak{K}_1$, we denote the class of all models in $\mathfrak{K}_0$ in which X is valid. Let $\mathfrak{K}$ be the class of all models with identity which belong to the class $\mathfrak{K}_1$. A class $\mathfrak{K}_0 \subset \mathfrak{K}$ is called $P_{\eta,\tau}$-*definable*, if there exists a set $X \subset P_{\eta,\tau}$ such that $\mathfrak{K}_0 = \mathfrak{K}(X)$.

Let α be any ordinal number. A class $\mathfrak{K}_0 \subset \mathfrak{K}$ is called $\alpha - P_{\eta,\tau}$-*definable*, if there exists a set $X \subset P_{\eta,\tau}$ such that $\mathfrak{K}_0 = \mathfrak{K}(X)$ and for all $p \in X$ $\overline{s(p)} < \aleph_\alpha$.

5 *). Since the set $P_{\eta,\tau}$ is closed with respect to the operations $\rightarrow$ and $'$, $\langle P_{\eta,\tau}, \rightarrow, ' \rangle$ is an algebra of the type $\langle 2, 1 \rangle$. Let $\boldsymbol{O} = \langle O, \rightarrow, ' \rangle$ be the subalgebra of $\langle P_{\eta,\tau}, \rightarrow, ' \rangle$ which is generated by U_τ. Obviously, $O \subset P_{\eta,\tau}$ and $\boldsymbol{O}$ is an absolutely free algebra of the type $\langle 2, 1 \rangle$ freely generated by U_τ. The elements of the algebra $\boldsymbol{O}$ are called O-formulae.

We shall now define certain operations which can be performed on subsets of O; we call them the *operations of consequence.*

Let X be any subset of O.

(5.1) $\boldsymbol{Cn}_O(X)$ *is the least subset* $X_1 \subset O$ *containing* X *and possessing the following properties*:

(a) *to* X_1 *belong all the* O*-formulae of one of the following forms*:

(a_1) $[p_1 \rightarrow (p_2 \rightarrow p_3)] \rightarrow [(p_1 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_3)]$,

(a_2) $p \rightarrow (p_1 \rightarrow p)$,

(a_3) $p \rightarrow p$,

(a_4) $(p \rightarrow p') \rightarrow p'$,

(a_5) $p' \rightarrow (p \rightarrow p_1)$,

(a_6) $p'' \rightarrow p$,

where p, p_1, p_2, p_3 *are arbitrary* O*-formulae.*

(b) *if* $p, p \rightarrow p_1 \in X_1$ *then* $p_1 \in X_1$.

(c) *for every* $\varphi \in W^N$ *if* $p \in X_1$ *then* $p_{(\varphi)} \in X_1$.

(5.2) $\boldsymbol{Cn}_{O=}(X)$ *is the least subset* $X_1 \subset O$ *containing* X *and possessing the properties* (a), (b), (c) *and the following properties*:

(i) *to* X_1 *belong all* O*-formulae of one of the following forms* **):

(i_1) $r_0(\vartheta, \vartheta)$,

(i_2) $r_0(\vartheta_0, \vartheta_1) \rightarrow r_0(\vartheta_1, \vartheta_0)$,

(i_3) $[r_0(\vartheta_0, \vartheta_1) \wedge r_0(\vartheta_1, \vartheta_2)] \rightarrow r_0(\vartheta_0, \vartheta_2)$,

(i_4) $[r_0(\vartheta_0, \delta_0) \wedge r_0(\vartheta_1, \delta_1)] \rightarrow [r_0(\vartheta_0, \vartheta_1) \equiv r_0(\delta_0, \delta_1)]$,

where $\vartheta, \vartheta_0, \vartheta_1, \vartheta_2, \delta_0, \delta_1$ *are arbitrary terms*;

(ii) *for every* $\sigma < \beta$ *if the equations* $r_0(\vartheta_\xi, \delta_\xi)$, $\xi < \alpha_\sigma$ *belong to* X_1, *then the equation*

$$r_0(f_\sigma(\vartheta_0, \vartheta_1, \ldots \vartheta_\xi \ldots)_{\xi<\alpha_\sigma}, f_\sigma(\delta_0, \delta_1, \ldots \delta_\xi \ldots)_{\xi<\alpha_\sigma})$$

belongs to X_1;

(iii) *for every* $\xi < \beta_1$ *if the equations* $r_0(\vartheta_\sigma, \delta_\sigma)$, $\sigma < \lambda_\xi$ *belong to* X_1, *then* $r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi} \equiv r_\xi(\delta_0, \delta_1, \ldots \delta_\sigma \ldots)_{\sigma<\lambda_\xi}$ *belongs to* X_1.

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*) This section contains essential results of my dissertation submitted to the Faculty of Mathematics, Physics and Chemistry of the Toruń University and accepted by that Faculty on 23. X. 1957.

**) In the sequel we shall write:

$p \vee q$ instead of $p' \rightarrow q$
$p \wedge q$ instead of $(p' \vee q')'$
$p \equiv q$ instead of $(p \rightarrow q) \wedge (q \rightarrow p)$

where p, q are arbitrary O-formulae.

We shall also use the consequences $\boldsymbol{Cn}_I(X)$ and $\boldsymbol{Cn}_{I=}(X)$. The consequence $\boldsymbol{Cn}_I(X)$ is based on the rules (a), (b). The consequence $\boldsymbol{Cn}_{I=}(X)$ is based on the rules (a), (b), (i), (ii), (iii). It is easy to verify that all those consequences have the following properties:

(5.3) $$X \subset \boldsymbol{Cn}(X) = \boldsymbol{Cn}(\boldsymbol{Cn}(X)) .$$

(5.4) If $X \subset X_1$, then $\boldsymbol{Cn}(X) \subset \boldsymbol{Cn}(X_1)$.

(5.5) There exists an O-formula p such that $\boldsymbol{Cn}((p)) = O$. The consequences $\boldsymbol{Cn}_I$ and $\boldsymbol{Cn}_O$ have a finite character, i. e.

(5.6) $$\boldsymbol{Cn}(X) = \sum_{\substack{Y \subset X \\ Y \text{ finite}}} \boldsymbol{Cn}(Y)$$ (finiteness property).

The consequences $\boldsymbol{Cn}_{I=}$ and $\boldsymbol{Cn}_{O=}$ have an infinite character, i. e.

(5.7) $$\boldsymbol{Cn}(X) = \sum_{\substack{Y \subset X \\ \overline{\overline{Y}} < \gamma_1}} \boldsymbol{Cn}(Y) \quad (\text{for } \boldsymbol{Cn} = \boldsymbol{Cn}_{O=} \text{ and } \boldsymbol{Cn} = \boldsymbol{Cn}_{I=}) .$$

We remark that the operations of consequence here considered map 2^O in 2^O. For the consequences given above various known methodological notions may be defined (definitions of such notions are contained in [1]). Now we shall consider Gödel's theorem which states:

"*Every consistent system has a model*".

(5.8) *Every consistent* O-system (*resp.* $O=$-system) *has a model.*

Proof. Let $X_1 \subset O$ be any consistent O-system (resp. $O=$-system). X_1 is also an I-consistent system. From (5.6) and (5.5) it follows that there exists an I-complete system $X \subset O$ which contains X_1. Lindenbaum's model

$$\boldsymbol{L}^{(X)} = \langle W, f_0, f_1, \ldots f_\sigma, \ldots; R_0^{(X)}, R_1^{(X)}, \ldots R_\xi^{(X)}, \ldots \rangle_{\sigma<\beta, \xi<\beta_1} ,$$

in which $\langle W, f_0, f_1, \ldots f_\sigma, \ldots \rangle_{\sigma<\beta}$ is algebra of terms and $R_\xi^{(X)}(\vartheta_0, \vartheta_1, \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}$ holds if, and only if, $r_\xi(\vartheta_0, \vartheta_1, \ldots \vartheta_\sigma \ldots)_{\sigma<\lambda_\xi}$ belongs to X, is a model of system X_1.

If we admit only the models with identity, then Gödel's theorem for $O=$-system fails as we shall see from the following example. Let $\mathfrak{K}_1$ be the class of all models with one operation f of the type ω_0 and with an infinite number of constants $a_n^0, a_n^1, b_n^0, b_n^1$, where $n = 0, 1, 2, \ldots$ and moreover with one binary relation r_0.

Let X be the set of all O-formulae of one of the following forms:

(A) $$r_0(a_n^0, b_n^0) \vee r_0(a_n^1, b_n^1) , \quad \text{where} \quad n = 0, 1, 2, \ldots$$

(B) $$[r_0(f(a_0^{t_0}, a_1^{t_1}, \ldots a_n^{t_n}, \ldots)_{n<\omega_0}, f(b_0^{t_0}, b_1^{t_1}, \ldots b_n^{t_n}, \ldots)_{n<\omega_0}]' ,$$

where $t_0, t_1, \ldots t_n, \ldots$ is an arbitrary sequence composed by 0 or 1.

It can be proved that

(5.9) *Every subset $X_0 \subset X$ which does not contain at least one formula of the form* (B) *has a model with identity, but the whole set X does not have a model with identity.*

From (5.9) and (5.7) it follows that

(5.10) $\boldsymbol{Cn}_{O=}(X)$ *is an $O=$-consistent system which does not have a model with identity.*

Moreover, the following theorems can be proved:

(5.11) *If $X_0 \subset O$ is a consistent $O=$-system, then X_0 has a model with identity if, and only if, there exists an I-complete system $X \subset O$ such that $X_0 \subset X$ and X is an $I=$-system.*

(5.12) *If $X_0 \subset O$ is a consistent $O=$-system which has the property*

(C) *if a disjunction $\varepsilon_1 \vee \varepsilon_2 \vee \ldots \vee \varepsilon_k$, where $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_k$ are equations, belongs to X_0, then an equation ε_i belongs to X_0,*

then system X_0 has a model with identity.

(5.13) *Let $X_0 \subset O$ be an $O=$-system and let Z be the set of all equations which do not belong to X_0.*

The system X_0 has a model with identity if, and only if, there exist two disjoint sets of equations E_1 and E_2 such that

(1) $$Z = E_1 + E_2$$

(2) *the set $X_0 + E_1 + E_2'$ is $I=$-consistent, where E_2' is the set of all negations of the equations in E_2.*

(5.14) *If a set $X_0 \subset O$ has a model with identity, then X_0 has a model with identity of a power equal or less than a power of algebra of terms.*

6. Let $\mathfrak{K}_0 \subset \mathfrak{K}$. We shall denote the class of all isomorphic images of models in $\mathfrak{K}_0$ and the class of all submodels of models in $\mathfrak{K}_0$ by $I(\mathfrak{K}_0)$ and $S(\mathfrak{K}_0)$. A model $\boldsymbol{M} \in \mathfrak{K}$ is generated by a set D, if the algebra $\boldsymbol{A}$ of model $\boldsymbol{M}$ is generated by D. By $G_\alpha(\boldsymbol{M})$, where α is any ordinal number, we denote the class of all submodels of $\boldsymbol{M}$ which are generated by the sets of a power less than $\aleph_\alpha$ *).

Let α be an arbitrary ordinal number greater than zero and let U_ξ be the set of atomic formulae corresponding to ξ. By α^* we shall denote an ordinal number such that $\aleph_{\alpha^*}$ is the least cardinal number greater than $\overline{\overline{U}}_\xi$, for all $\xi < \alpha$. Let $v_0, v_1, \ldots v_n$ be variables. By U_n' we shall denote the set of all atomic formulae the supports of which are subsets of set $(v_0, v_1, \ldots v_n)$. By 0^* we shall denote an ordinal number such that $\aleph_{0^*}$ is the least cardinal number greater than $\overline{\overline{U}}_n'$, for all $n < \omega_0$.

*) If M has only relations, then $G_\alpha(\boldsymbol{M})$ denotes the class of all submodels of M with a power less than α.

The following theorems can be proved:

(6.1) *Let* η, α, τ *be arbitrary ordinal numbers such that* $\alpha \leqslant \tau+1$ *and* $\eta \geqslant \alpha^*$.

A class $\mathfrak{K}_0 \subset \mathfrak{K}$ *is* $\alpha - P_{\eta,\tau}$*-definable if, and only if,* $\mathfrak{K}_0$ *fulfills the following condition*:

(j) *for every* $\boldsymbol{M} \in \mathfrak{K}$ *if* $G_\alpha(\boldsymbol{M}) \subset I(S(\mathfrak{K}_0))$, *then* $\boldsymbol{M} \in \mathfrak{K}_0$.

(6.2) *Let* α *be any ordinal number such that* $\aleph_\alpha$ *is regular or* Δ_1*-regular and* $\aleph_\alpha \geqslant \overline{\beta} \cdot \overline{\beta}_1 \cdot \overline{\gamma}_1$.

Let, moreover, $\tau \geqslant \alpha$.

(6.2.1) *A class* $\mathfrak{K}_0 \subset \mathfrak{K}$ *is* $P_{\alpha+1,\tau}$*-definable if, and only if,* $\mathfrak{K}_0$ *fulfills the following condition*:

(jj) *for every* $\boldsymbol{M} \in \mathfrak{K}$ *if* $G_{\alpha+1}(\boldsymbol{M}) \subset I(S(\mathfrak{K}_0))$, *then* $\boldsymbol{M} \in \mathfrak{K}_0$

(6.2.2) *A class* $\mathfrak{K}_0 \subset \mathfrak{K}$ *is* $\alpha - P_{\alpha+1,\tau}$*-definable if, and only if,* $\mathfrak{K}_0$ *fulfills condition* (j) (see (6.1)).

The proof of Theorem (6.1) is analogous to the proof of Tarski's theorem 1.2 in [2] and theorems 1 and 2 in [3]. The theorem (6.2) follows from (6.1), (4.2) and the definition of number α^*.

Finally we remark that from the Theorems (6.1) and (6.2) follow Tarski's theorems: theorem 1.2 in [2] and theorems 1 and 2 in [3].

NICHOLAS COPERNICUS UNIVERSITY, TORUŃ

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

MATHÉMATIQUE

Sur les fonctionnelles linéaires dans des espaces vectoriels semiordonnés avec application à représentation des fonctionnelles par des intégrales

par

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Présenté par W. ORLICZ le 17 mai, 1958

G. Fichtenholz [4] a analysé une convergence de suites $x_n(t)$ de fonctions mesurables et bornées:

$$x_n \xrightarrow{\gamma} x \quad \textit{lorsque} \begin{cases} \sup\limits_{n=1,2,\ldots} \operatorname{ess\,sup}\limits_{\langle 0,1\rangle} |x_n(t)| < \infty, \\ x_n(t) \textit{ convergent en mesure vers } x_0(t). \end{cases}$$

La convergence γ est du type $\mathcal{L}^*$ de Fréchet (dans l'espace L^∞ des fonctions mesurables, essentiellement bornées).

Dans cette note je considère une généralisation de cette conception de convergence. La méthode présentée se rattache aux travaux de A. Alexiewicz et de W. Orlicz concernant la théorie des espaces linéaires munis de deux normes non-équivalentes; pourtant, mon traitement de la matière rappelle plutôt celui de Fichtenholz.

1. Le scheme général

Soit X un treillis vectoriel normé *) dans lequel une notion de limite $\mathcal{L}$ au sens de Fréchet est définie de la manière:

$$(1) \qquad x_n \xrightarrow[l]{} x_0,\ y_n \xrightarrow[l]{} y_0,\ \lambda_n \to \lambda_0 \Rightarrow x_n + \lambda_n y_n \xrightarrow[l]{} x_0 + \lambda_0 y_0,$$

$$(2) \qquad x_n \geqslant 0\ (n = 1, 2, \ldots),\ x_n \xrightarrow[l]{} x_0 \Rightarrow x_0 \geqslant 0,$$

$$(3) \qquad x_n \xrightarrow[l]{} 0,\ 0 \leqslant y_n \leqslant x_n \Rightarrow y_n \xrightarrow[l]{} 0,$$

*) Nous admettons, comme d'habitude, que les implications

$$0 \leqslant x \leqslant y \Rightarrow \|x\| \leqslant \|y\| \quad \text{et} \quad x \wedge y = 0 \Rightarrow \|x+y\| = \|x-y\|$$

sont valables (cf. la définition de G. Birkhoff [3] p. 246).

(4) $$\|x_n\| \underset{l}{\to} 0 \quad \Rightarrow \quad x_n \underset{l}{\to} 0\,,$$

(5) $$x_n \underset{l}{\to} 0 \quad \Rightarrow \quad |x_n| \underset{l}{\to} 0 \text{ (ou bien } x_n^+ \underset{l}{\to} 0 \text{ et } x_n^- \underset{l}{\to} 0)\,.$$

Les axiomes (1)-(5) sont satisfaits dans ces deux cas typiques. Le premier cas se présente lorsque la convergence l est métrisable par une norme $\|\cdot\|^*$ („la norme faible") telle que $|x_n| \to 0 \Rightarrow \|x_n\|^* \to 0$ et $|x| \leqslant |y| \Rightarrow \|x\|^* \leqslant \|y\|^*$; alors $\langle X, \|\cdot\|, \|\cdot\|^*\rangle$ est dit *treillis vectoriel binormé*. Dans le deuxième cás x_n sont des fonctions et $x_n \underset{l}{\to} 0$ désigne la convergence de la suite $x_n(t)$ en chaque point t (ou bien presque partout). Cependant, la convergence faible d'éléments d'un treillis normé et la convergence faible des fonctionnelles linéaires ne satisfont pas nécessairement à la condition (5) *).

Etant données la norme $\|\cdot\|$ et la convergence l, nous définissons

$$x_n \underset{l}{\to} x_0 \quad \textit{lorsque} \quad \sup_{n=1,2,\ldots} \|x_n\| < \infty \quad \textit{et} \quad x_n \underset{l}{\to} x_0\,.$$

Désignons par Ξ l'espace conjugué à $\langle X, \|\cdot\|\rangle$ (c'est-à-dire l'ensemble des fonctionnelles linéaires, continues par rapport à la norme $\|\cdot\|$). Ξ constitue un treillis de Banach muni de la norme $\|\xi\| = \sup\{\xi(x) : |x| \leqslant 1\}$ et de l'ordre naturel $\xi \geqslant 0$ *lorsque* $\xi(x) \geqslant 0$ *pour* $x \geqslant 0$; $\xi \vee \eta$ et $\xi \wedge \eta = -[(-\xi) \vee (-\eta)]$ sont déterminés (pour $x \geqslant 0$) par les formules connues de Riesz [11]:

$$(\xi \vee \eta)(x) = \sup\{\xi(x_1) + \eta(x_2) : \quad x_1 \geqslant 0,\ x_2 \geqslant 0, \quad x_1 + x_2 = x\}\,.$$

Désignons par Ξ_l l'ensemble des fonctionnelles l-linéaires (continues par rapport à la convergence l) et par Ξ_γ l'ensemble des celles γ-linéaires. Evidemment $\Xi_l \subset \Xi_\gamma \subset \Xi$ **).

Théorème 1. *Les ensembles* Ξ_l *et* Ξ_γ *sont des sous-treillis linéaires de* Ξ, *de plus* Ξ_γ *est fermé* ***):

(6) $$\xi, \eta \in \Xi_l \ \Rightarrow\ \xi \vee \eta \in \Xi_l,\ \xi \wedge \eta \in \Xi_l$$

(7) $$\xi, \eta \in \Xi_\gamma \ \Rightarrow\ \xi \vee \eta \in \Xi_\gamma,\ \xi \vee \eta \in \Xi_\gamma$$

(8) $$\xi_n \in \Xi_\gamma (n = 1, 2, \ldots),\ \lim \|\xi_n - \xi_0\| = 0 \ \Rightarrow\ \xi_0 \in \Xi_\gamma$$

Démonstration. Il s'agit de demontrer en vertu des identités

$$\xi \vee \eta = (\xi - \eta)_+ + \eta, \qquad \xi \wedge \eta = -(\eta - \xi)_+ + \eta$$

*) Dans l'espace l^2, par exemple, la condition (5) est satisfaite, tandis que dans L^2 et pour les fonctionnelles définies dans $C\langle 0, 1\rangle$ elle ne l'est pas.

**) On n'y exclue pas que Ξ_l (ou Ξ_γ) se compose de la seule fonctionnelle 0.

***) Pour le cas où la convergence l est déterminée par une norme $\|\ \|^*$, la condition (8) fut démontrée par W. Orlicz et V. Pták, voir [10], p. 57.

(voir [3], p. 215), que pour chaque $\xi \in \Xi_\gamma$ la fonctionnelle

$$\xi_+(x) = \begin{cases} \sup\{\xi(y) : 0 \leqslant y \leqslant x\} \text{ pour } x \geqslant 0 \\ \xi_+(x^+) - \xi_+(x^-) \text{ pour les autres } x \end{cases}$$

appartient à Ξ_l.

Admettons, que $x_n \underset{l}{\to} 0$, alors $x_n^+ \underset{l}{\to} 0$ et $x_n^- \underset{l}{\to} 0$ (d'après (3) et (5)). Pour tout n il existe $y_n \in X$ tel que $0 \leqslant y_n \leqslant x_n^+$ et $|\xi_+(x_n^+) - \xi(y_n)| \leqslant \frac{1}{n}$. D'après (3) il vient $y_n \underset{l}{\to} 0$, donc $\xi(y_n) \underset{l}{\to} 0$. De plus,

$$|\xi_+(x_n^+)| \leqslant |\xi_+(x_n^+) - \xi(y_n)| + |\xi(y_n)| \leqslant \frac{1}{n} + |\xi(y_n)|,$$

ce qui prouve que $\lim \xi(x_n^+) = 0$. De façon analogue on constate que $\lim \xi_+(x_n^-) = 0$, d'où $\lim \xi_+(x_n) = 0$. La condition (7) resulte de (6) (la convergence γ satisfait aussi à (1)-(5)). Afin de prouver (8), supposons que $\xi_n \in \Xi_\gamma$, $\xi_0 \in \Xi$, $\lim \|\xi_n - \xi_0\| = 0$ et $x_n \underset{\gamma}{\to} 0$. Alors

$$|\xi_0(x_n)| \leqslant |\xi_m(x_n)| + |\xi_m(x_n) - \xi_0(x_n)| \leqslant |\xi_m(x_n)| + \|\xi_m - \xi_0\| \cdot \sup \|x_n\|$$

entraîne $\lim \xi_0(x_n) = 0$, d'où $\xi_0 \in \Xi_\gamma$.

2. Applications: représentation des fonctionnelles par des intégrales

2.1. Soit E un ensemble fixé, $\mathfrak{A}$ un corps de sous-ensembles de E (algèbre de Boole), $\mathbf{R}$ un σ-idéal contenu dans $\mathfrak{A}$ (c'est-à-dire famille héréditaire et σ-additive). Désignons par X_0 la classe de toutes combinaisons linéaires des fonctions caractéristiques des ensembles de $\mathfrak{A}$

$$x(t) = \sum_{k=1}^{n} a_k \chi_{A_k}(t), \quad A_i \in \mathfrak{A}, \tag{9}$$

et par X la classe des limites uniformes des suites $x_n(t)$ de fonctions (9).

Dans X nous introduisons la pseudonorme

$$\|x\| = \sup_{t \in E}{}_{\mathbf{R}} |x(t)|,$$

où $\sup_A{}_{\mathbf{R}} y(t)$ désigne le plus petit nombre α tel que l'ensemble

$$\{t \in A : x(t) > \alpha\}$$

appartienne à $\mathbf{R}$, et nous identifions les fonctions R-équivalentes (égales $\mathbf{R}$-presque partout = égales hors d'un certain ensemble de $\mathbf{R}$); alors l'espace $X/\|\cdot\|$ est un espace de Banach. Désignons le par $\Lambda = \Lambda(E, \mathfrak{A}, \mathbf{R})$.

La forme générale des fonctionnelles linéaires définies dans A est la suivante *):

Toute fonctionnelle $\xi(x)$ définie dans $A(E, \mathfrak{A}, \mathbf{R})$ est de la forme

$$\xi(x) = \int_E x(t)\,d\mu\,, \tag{10}$$

où μ est une fonction additive à variation bornée, définie pour les ensembles de $\mathfrak{A}$, qui s'annule sur les ensembles de $\mathbf{R}$. De plus

$$\|\xi\| = \|\xi_+\| + \|\xi_-\| = \mu_+(E) + \mu_-(E) = |\mu|(E) = \operatorname*{Var}_E \mu\,.$$

Pour que μ soit non-négative, il faut et il suffit que $\xi \geqslant 0$. La *mesure* μ est déterminé par ξ de façon biunivoque par la formule

$$\mu(A) = \mu_+(A) - \mu_-(A) = \sup\{\xi\chi_B\colon B \subset A,\ B \in \mathfrak{A}\} - \inf\{\xi\chi_B\colon B \subset A,\ B \in \mathfrak{A}\} \tag{11}$$

(en particulier, si $\xi \geqslant 0$, il vient $\mu(A) = \xi\chi_A$). L'intégrale (10) est définie de la manière suivante:

$$\begin{cases} \int_E x(t)\,d\mu = \int_E x(t)\,d\mu_+ - \int_E x(t)\,d\mu_-\,, \\ \text{où (pour } \nu \geqslant 0) \int_E x(t)\,d\nu = \inf_{\{A_i\}} \sum_{i=1}^{n} \alpha_i \nu(A_i) = \sup_{\{A_i\}} \sum_{i=1}^{n} \beta_i \nu(A_i)\,, \\ \{A_i\} \text{ parcourt toutes les partitions finies de } E,\ \alpha_i = \sup_{A_i \mathbf{R}} x(t) \\ \text{et } \beta_i = \inf_{A_i \mathbf{R}} x(t)\,. \end{cases} \tag{12}$$

Généralement la fonction (11) n'est pas σ-additive **).

Introduisons dans A la convergence $\mathbf{R}$-presque partout

$x_n \underset{l}{\to} x_0$ *lorsqu'il existe* $A \in \mathbf{R}$ *tel que* $x_n(t) \to x_0(t)$ *pour* $t \in E - A$

et la convergence $x_n \underset{l}{\to} x_0$ comme auparavant.

Théorème 2. *Soit ξ une fonctionnelle linéaire, définie dans l'espace $A(E, \mathfrak{A}, \mathbf{R})$. La condition nécessaire et suffisante pour que la fonction μ correspondant à ξ soit σ-additive est que ξ soit γ-continue* ***).

*) Les principes de cette méthode sont dues à Banach ([2], p. 218), mais cet auteur ne désignait pas l'expression (12) comme *intégrale*. Pour les espaces L^∞, $\mathbf{m}$ (des suites bornées) et pour quelques analogues le théorème ci-dessus fut démontré par G. Fichtenholz et L. Kantorovitsch [5] et (indépendamment) par T. H. Hildebrandt [6]. De façon plus générale il fut établi par K. Yosida et E. Hewitt ([12], p. 53).

**) Par exemple, considérons le cas $E = \langle 0, 1\rangle$, $\mathfrak{A}$ = le corps des ensembles mesurables, $\mathbf{R}$ = l'idéal des ensembles de mesure nul. Alors $A(E, \mathfrak{A}, \mathbf{R})$ coïncide avec l'espace L^∞ des fonctions mesurables, essentiellement bornées et chaque fonctionnelle linéaire et multiplicative ($\neq 0$) correspond à une mesure qui n'est pas σ-additive.

***) Pour l'espace L^∞ ce théorème est dû à G. Fichtenholz [4], pour l'espace $\mathbf{m}_\gamma (\mathbf{m}_\gamma = A(N, 2^N, \{0\}))$, où N désigne l'ensemble des nombres naturels) il est dû à A. Alexiewicz ([1], p. 53).

Démonstration. La suffisance résulte du théorème de Lebesgue sur l'intégration des suites bornées. Pour montrer la nécessité appliquons le théorème 1: chaque fonctionnelle γ-linéaire peut être représentée comme la difference de deux fonctionnelles γ-linéaires positives. Soit $\xi \in \Xi_\gamma$, $\xi \geqslant 0$ et $A_n \in \mathfrak{A}$, $A \in \mathfrak{A}$ tels que $A_i \cap A_j = 0$ pour $i \neq j$ et $A = \bigcup_{n=1}^{\infty} A_n$. Désignons par $x_n = \chi_{B_n}$ la fonction caractéristique de l'ensemble $B_n = A_1 \cup \ldots \cup A_n$. Evidemment, $\|x_n\| \leqslant 1$ et $x_n(t) \to x(t) = \chi_A(t)$ pour tous les $t \in E$, donc $x_n \underset{\gamma}{\to} x$. Par conséquent

$$\mu(A) = \xi\chi_A = \lim \xi\chi_{B_n} = \lim \xi(\chi_{A_1} + \ldots + \chi_{A_n}) = \sum_{n=1}^{\infty} \xi\chi_{A_n} = \sum_{n=1}^{\infty} \mu(A_n)\,.$$

2.2. Soit E un espace topologique complètement régulier et σ-compact (c'est-à-dire représentable comme la réunion $E = \bigcup_{n=1}^{\infty} E_n$ d'une famille dénombrable d'ensembles compacts $E_n \subset E_{n+1}$). Désignons par $C^*(E)$ l'espace des fonctions réelles, continues et bornées dans E, muni de la norme $\|x\| = \sup_E |x(t)|$ et la convergence

$$x_n \underset{l}{\to} x_0 \quad \textit{lorsque} \quad x_n(t) \to x_0(t) \textit{ uniformément sur tout } E_n\,.$$

Selon S. Kakutani ([7], p. 1006 et 1012) toute fonctionnelle linéaire, définie dans $C^*(E)$ est de la forme

$$\xi(x) = \int_{\beta(E)} \bar{x}(t)\,d\mu\,,$$

où μ est une fonction σ-additive définie dans le corps des ensembles boreliens de βE et $\bar{x}(t)$ est la fonction prolongée de $x(t)$ (β désigne la compactification maximale au sens de Stone et Čech). Si $\xi \geqslant 0$, la mesure μ est déterminée pour G ouverts par la formule

$$\mu(G) = \sup\{\xi(x):\ \ \|x\| \leqslant 1,\ x(t) = 0 \textit{ pour } t \in \beta E - G\}\,. \tag{13}$$

Théorème 3. *Soit $X = C^*(E)$ et $\xi \in X^*$. Pour que la fonction μ qui correspond à ξ s'annule hors de E (autrement dit, pour que $\operatorname{Var}_{\beta E - E} \mu = 0$), il faut et il suffit que ξ soit γ-linéaire* *).

Démonstration. La suffisance étant évidente en vertu du théorème cité de Lebesgue, il nous faut prouver la nécessité. Soit $\xi \in \Xi_\gamma$. Selon le th. 1 nous pouvons nous borner au cas $\xi \geqslant 0$. D'après (13) il

*) De là on déduit que toute fonctionnelle γ-linéaire, définie dans $C^*(E)$ est de la forme $\int_E x(t)\,d\mu$. Ce théorème, publié par J. Mařík ([8], p. 90 et 92), est une généralisation du théorème de J. Musielak et W. Orlicz ([9], p. 219). J'ai taché de prouver ce même théorème encore d'une autre manière.

existe pour $n = 1, 2, \ldots$ une fonction $x_n \in C^*(E)$ telle que $\|x_n\| \leqslant 1$, $x_n(t) = 0$ pour $t \in E_n$ et $|\xi(x_n) - \mu(\beta E - E_n)| < \frac{1}{n}$. Evidemment $x_n \underset{\gamma}{\to} 0$, donc $\xi x_n \to 0$ et finalement $\mu(\beta E - E) = \lim_n \mu(\beta E - E_n) = 0$.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

The Möller Scattering of Arbitrarily Polarized Electrons

by

R. RĄCZKA and A. RĄCZKA

Presented by L. INFELD on May 5, 1958

In this paper the general covariant formula for the cross-section of the Möller scattering is obtained. Then, the special case of transversal polarization is discussed. We find the experimental conditions that are to be satisfied if the Möller scattering is to be used for the measurement of the transversal component of the polarization vector. The relative magnitude of polarization effects is discussed as function of the scattering angle and energy.

General formula for the cross-section

The process of scattering of polarized electrons on polarized electrons is described in the first approximation by the following Feynmann diagrams:

The four-vector $p_1 = (\vec{p}_1, \varepsilon_1)$ is the momentum of the incident electrons, $p_2 = (\vec{p}_2, \varepsilon_2)$ is the momentum of the target electrons, and $p_1' = (\vec{p}_1', \varepsilon_1')$, $p_2' = (\vec{p}_2', \varepsilon_2')$ are the momenta of scattered electrons.

Let us calculate the matrix element corresponding to a transition from the initial state $|i\rangle = |p_1, S_1, p_2, S_2\rangle$ to the final state $\langle f| = \langle p_1', S_1', p_2', S_2'|$, where the unit four-pseudovector [1] $S = \left(\vec{l} + \frac{(\vec{l}\,\vec{p})}{m(\varepsilon+m)}\vec{p}, \frac{(\vec{l}\,\vec{p})}{m}\right)$ is the covariant polarization vector of electrons. From Feynmann-Dyson's formalism we obtained for this matrix element the following expression:

$$S_{if} = \frac{1}{i} \cdot \left(\frac{em}{2\pi}\right)^2 \cdot \frac{\delta(p_1 + p_2 - p_1' - p_2')}{\sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_1' \varepsilon_2'}} \cdot \langle f | M | i \rangle , \tag{1}$$

where

$$(2)\qquad \langle f|M|i\rangle = \frac{\bar u(p_1', S_1')\gamma^\nu u(p_1, S_1)\bar u(p_2', S_2')\gamma_\nu u(p_2, S_2)}{(p_1-p_1')^2} + \\ - \frac{\bar u(p_2', S_2')\gamma^\mu u(p_1, S_1)\bar u(p_1', S_1')\gamma_\mu u(p_2, S_2)}{(p_1-p_2')^2} = a-\beta\,.$$

The cross-section for the process under consideration is given by

$$(3)\qquad \sigma = 4\left(\frac{em}{2\pi}\right)^4 \cdot \int \frac{d_3\vec p_1' d_3\vec p_2'}{\varepsilon_1'\varepsilon_2'\sqrt{(p_1p_2)^2-m^4}} \cdot \delta(p_1+p_2-p_1'-p_2')|\langle f|M|i\rangle|^2\,,$$

where

$$(4)\qquad |\langle f|M|i\rangle|^2 = (a-\beta)^+(a-\beta) = a^+a+\beta^+\beta-(a^+\beta+\beta^+a)\,.$$

The expression (4) includes the products of the spinors $u(p_1, S_1)$, $u(p_2, S_2)$, $u(p_1', S_1')$, $u(p_2', S_2')$ of the free electrons. These products of spinors can be replaced by the products of the projection operator of polarization $\Gamma(S) = \frac{1}{2}(1+\gamma_5\gamma_\mu S^\mu)$ and the projection operator of the energy sign $\Lambda(p) = 1/2m(m-ip^\mu\gamma_\mu)$ in the following way:

$$u(p, S)\bar u(p, S) = \Gamma(S)[u(p, S)\bar u(p, S)+u(p, -S)\bar u(p, -S)] = \Gamma(S)\cdot\Lambda_-(p)\,.$$

Using the above relation and the condition of the normalization of spinors we obtain:

$$(5)\qquad \begin{aligned} a^+a &= t^2 Tr[\gamma_\mu\Gamma(S_2)\Lambda_-(p_2)\gamma_\nu\Lambda_-(p_2')]\cdot Tr[\gamma^\mu\Gamma(S_1)\Lambda_-(p_1)\gamma^\nu\Lambda_-(p_1')]\,,\\ \beta^+\beta &= k^2 Tr[\gamma_\nu\Gamma(S_2)\Lambda_-(p_2)\gamma_\mu\Lambda_-(p_1')]\cdot Tr[\gamma^\nu\Gamma(S_1)\Lambda_-(p_1)\gamma^\mu\Lambda_-(p_2')]\,,\\ a^+\beta+\beta^+a &= 2tk\{Tr[\gamma_\mu\Gamma(S_2)\Lambda_-(p_2)\gamma_\nu\Lambda_-(p_2')\gamma^\mu\Gamma(S_1)\Lambda_-(p_1)\gamma^\nu\Lambda_-(p_1')]+\\ &\quad + Tr[\gamma^\nu\Gamma(S_1)\Lambda_-(p_1)\gamma^\mu\Lambda_-(p_2')\gamma_\nu\Gamma(S_2)\Lambda_-(p_2)\gamma_\mu\Lambda_-(p_1')]\}\,. \end{aligned}$$

After rather wearisome calculations we obtain the following general covariant formula for the cross-section:

$$d\sigma = d\sigma_M + d\sigma_P = 4r_0^2\frac{m^2\beta_1'\varepsilon_1' d\Omega}{\sqrt{(p_1p_2)^2-m^4}}\cdot\frac{d\varepsilon_1'}{d(\varepsilon_1'+\varepsilon_2')}\cdot(X_M+X_P)\,,$$

$$(6)\qquad \begin{aligned} X_M &= t^2[2m^4+2m^2(p_1p_1')+(p_1p_2)^2+(p_1p_2')^2]+k^2[2m^4+2m^2(p_1p_2')+\\ &\quad +(p_1p_2)^2+(p_1p_1')^2]+tk[4m^2(p_1p_2)+2(p_1p_2)^2]\,,\\ X_P &= -m^2t^2[(S_1S_2)(p_1-p_1')^2+(S_1p_1')(S_2p_2')]-m^2k^2[(S_1S_2)(p_1-p_2')^2+\\ &\quad +(S_1p_2')(S_2p_1')]-tk[2(S_1p_2)(S_2p_1')(p_1p_1')+2(S_1p_1')(S_2p_1)(p_2p_1')+\\ &\quad -2(S_1p_1')(S_2p_1')(p_1p_2)-2(S_1S_2)(p_2p_1')(p_1p_1')+m^2(S_1p_2)(S_2p_1)+\\ &\quad +m^2(S_1p_2)(S_2p_1')+m^2(S_1p_1')(S_2p_1)-4m^2(S_1S_2)(p_1p_2)]\,, \end{aligned}$$

where $t = \dfrac{1}{\sqrt{2}(p_1-p_1')^2}$, $k = \dfrac{1}{\sqrt{2}(p_1-p_2')^2}\cdot d\sigma_M$ is the usual Möller cross-section without the polarization, and $d\sigma_P$ denotes the polarization dependent part of the cross-section.

In the centre of mass system the following relations are found:

$$\vec{p}_1 = -\vec{p}_2 = \vec{p}, \quad \vec{p}_1' = -\vec{p}_2' = \vec{p}', \quad |\vec{p}| = |\vec{p}'| = \beta\varepsilon,$$

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_1' = \varepsilon_2', \quad \varepsilon = m\gamma = \frac{m}{\sqrt{1-\beta^2}},$$

$$(\vec{p}\,\vec{p}') = m^2(\gamma^2-1)\cos\vartheta, \quad \sqrt{(p_1p_2)^2 - m^4} = 2\beta\gamma^2 m^2,$$

and the cross-section can be written in terms of γ and ϑ as follows:

$$(7) \quad d\sigma = \frac{r_0^2}{4\gamma^2}\cdot\frac{d\Omega}{m^2(\gamma^2-1)\sin^4\vartheta}\{m^2[4(2\gamma^2-1)^2-(8\gamma^4-4\gamma^2-1)\sin^2\vartheta+$$
$$+(\gamma^2-1)^2\sin^4\vartheta] + m^2(\vec{l}_1\vec{l}_2)[(3-4\gamma^2)\sin^2\vartheta+(\gamma^2-1)^2\sin\vartheta^4]+$$
$$+2(\vec{l}_1\vec{p}')(\vec{l}_2\vec{p}')(1-\gamma^2\sin^2\vartheta)+2(\vec{l}_1\vec{p})(\vec{l}_2\vec{p})[1-(5\gamma^2-2\gamma-2)\sin^2\vartheta+$$
$$+(\gamma-1)^2\sin^5\vartheta]+2\cos\vartheta[(\vec{l}_1\vec{p}')(\vec{l}_2\vec{p})+(\vec{l}_1\vec{p})(\vec{l}_2\vec{p}')[-1+$$
$$+\gamma(\gamma-1)\sin^2\vartheta]\}.$$

This expression is valid for arbitrary initial polarization $\vec{l}_1$ and $\vec{l}_2$. In the particular case of longitudinal polarization discussed by A. Bincer [2] our results are identical with his.

The transversal polarization

Let us discuss the possibility of using the Möller scattering for measuring the transversal component of the polarization vector $\vec{l}_1$. We can put in the laboratory system:

$$(8) \quad \text{a) } S_2 = (\vec{l}_2, 0), \quad \text{b) } (S_2p_1) = (\vec{l}_2, \vec{p}_1) = 0, \quad \text{c) } (S_2p_1') = (\vec{l}_2\vec{p}_1') = 0.$$

Assumption (8a) means that the target electrons are at rest. Their polarization vector $\vec{l}_2$ can be fixed by the direction of the magnetic field (e. g. in the ferromagnetic foil). Then condition (8b) means that the polarization vector $\vec{l}_2$ of the target electrons is perpendicular to the beam of incident electrons. The meaning of (8c) is that the measurement of the momenta of scattered electrons $\vec{p}_1'$, $\vec{p}_2'$ ought to be performed in a plane perpendicular to the vector $\vec{l}_2$ (see Fig. 1).

Because of (8) cross-section (6) has now the following much simpler form:

$$(9) \quad d\sigma = d\sigma_M + d\sigma_P = d\sigma_M + (S_1S_2)\,d\sigma_S,$$

where

$$(10) \quad d\sigma_S = 4r_0^2\,\frac{m^2\beta_1'\varepsilon_1'\,d\Omega}{\sqrt{(p_1p_2)^2-m^4}}\cdot\frac{d\varepsilon_1'}{d(\varepsilon_1'+\varepsilon_2')}\cdot$$
$$\cdot\left[\frac{-m^2}{2(p_1-p_1')^2}-\frac{m^2}{2(p_1-p_2')^2}-\right.$$
$$\left.-\frac{(p_2p_1')(p_1p_1')+2m^2(p_1p_2)}{(p_1-p_1')^2(p_1-p_2')^2}\right].$$

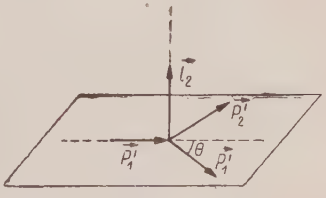

Fig. 1. Angular relations in the case of measurement of the transversal component of the polarization vector in the laboratory system

We see that, by virtue of (8), the polarization vectors S_1 and S_2 enter the cross-section only through the scalar product $(S_1 \cdot S_2)$.

In order to find the best conditions for the measurement of the polarization vector let us investigate the ratio:

$$\varkappa = \frac{d\sigma_P}{d\sigma_M} = (S_1 S_2)\frac{d\sigma_S}{d\sigma_M}. \tag{11}$$

In the laboratory system we have now $(S_1 \cdot S_2) = (\vec{l}_1 \cdot \vec{l}_2)$ and the expression (11) becomes:

$$\varkappa = (\vec{l}_1 \vec{l}_2)\,\frac{\frac{1}{4} + \frac{1-2\gamma}{(\gamma-1)^2(1-x^2)}}{\frac{\gamma^2}{(\gamma-1)^2}\cdot\left[\frac{4}{(1-x^2)^2} - \frac{3}{1-x^2} + \left(\frac{\gamma-1}{2\gamma}\right)^2\left(1+\frac{4}{1-x^2}\right)\right]}, \tag{12}$$

where

$$1-x^2 = \frac{2(\gamma+1)\sin^2 2\theta}{[2+(\gamma-1)\sin^2\theta]^2}, \tag{13}$$

and θ is the angle between $\vec{p}_1$ and $\vec{p}_1'$ in that system. In Fig. 2 the ratio $d\sigma_S/d\sigma_M$ is plotted against the angle of measurement for different values of β. The curves show strong maxima in the vicinity of $\theta = \pi/4$ which,

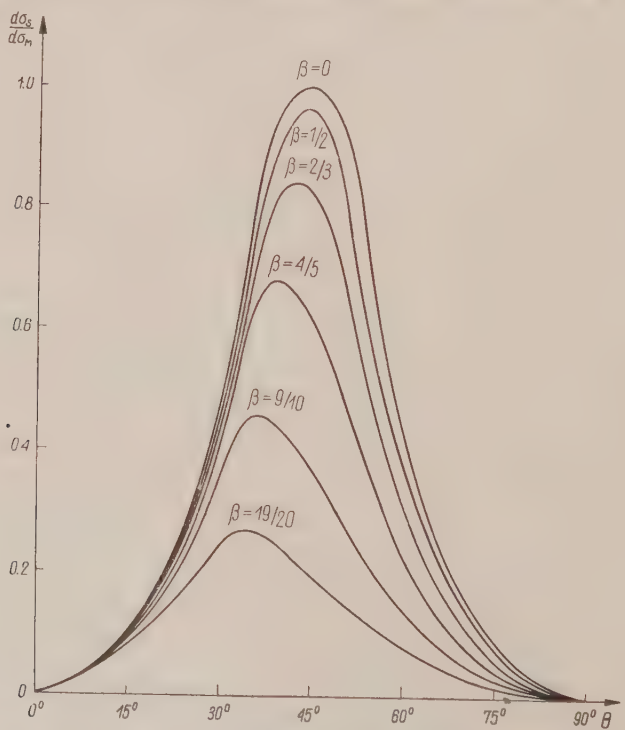

Fig. 2. The ratio $d\sigma_S/d\sigma_M$ against the scattering angle θ for different values of β

with increasing β, move towards smaller angles. Besides, one sees that the polarization effects increase with decreasing β.

To calculate the transversal component of the polarization vector $\vec{l}_1$ we find it convenient to introduce the ratio:

$$a = \frac{d\sigma_M + (S_1 S_2)\, d\sigma_S}{d\sigma_M - (S_1 S_2)\, d\sigma_S} = \frac{d\sigma_M + (\vec{l}_1 \vec{l}_2)\, d\sigma_S}{d\sigma_M - (\vec{l}_1 \vec{l}_2)\, d\sigma_S}, \tag{14}$$

which is the ratio of the cross-sections for two opposite spin orientations of the target electrons under the assumption that both target and beam electrons are completely polarized. In the case of partial polarizations of the incident electrons and target electrons, the $\vec{l}$'s which occur in the definition of S_1, S_2 are to be replaced by:

$$\vec{l}_1' = c_1 \vec{l}_1, \qquad \vec{l}_2' = c_2 \vec{l}_2, \tag{15}$$

where c_1 and c_2 denote the polarization degrees and the vectors $\vec{l}_1$ and $\vec{l}_2$ are unit vectors pointing in the direction of the average polarization of the electrons beam. Then a is the value which can be found experimentally.

We can now calculate $(\vec{l}_1, \vec{l}_2)$ from (14). We find

$$(\vec{l}_1' \vec{l}_2') = c_1 c_2 (\vec{l}_1 \vec{l}_2) = \frac{a-1}{a+1} \cdot \frac{d\sigma_M}{d\sigma_S} = \tag{16}$$

$$= \frac{a-1}{a+1} \cdot \frac{\dfrac{\gamma^2}{(\gamma-1)^2} \cdot \left[\dfrac{4}{(1-x^2)^2} - \dfrac{3}{1-x^2} - \left(\dfrac{\gamma-1}{2\gamma} \right)^2 \left(1 + \dfrac{4}{1-x^2} \right) \right]}{\dfrac{1}{4} + \dfrac{1-2\gamma}{(\gamma-1)^2 (1-x^2)}}.$$

When the constants c_1 and c_2 are known the transversal components of $\vec{l}_1'$ can be obtained from two measurements of a for two independent directions of $\vec{l}_2'$.

A paper by G. W. Ford and C. J. Mullin [3] dealing with a similar problem has recently been published. The general expressions for the cross-section given by these authors (formulae (4), (5) of [3]) differ in some respects from our formula (6). Contrary to the statement made by Ford and Mullin (footnote (7) on page 479 [3]), their results also differ from those of A. Bincer [2]. Moreover, it can be demonstrated that, for some other particular cases, e. g.:

$$(S_2 p_1) = 0, \qquad (S_2 p_1') = 0, \ \beta \to 0, \qquad \theta = \tau/4, \qquad (S_1 S_2) = 1,$$

their cross-section (4), (5) becomes negative showing that their calculations are not correct.

The authors wish to thank Dr. J. Werle, under whose direction this work was carried out, for suggesting the topic, for helpful discussions and critical remarks.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

Excited Levels of Nuclei with Mixed Configurations. II. Numerical Example

by

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Presented by L. INFELD on May 7, 1958

Introduction

In the first part of this work [1] *) a method of calculation of the nuclear excited levels in the intermediate mass region was presented. The method consisted in the diagonalization of the operator of residual interactions between all external nucleons. The second part aims at illustrating the previous considerations with a simple numerical example.

Let us consider the even-even nuclei with 4 external nucleons: 2 protons in the state $(1p_{3/2})$ and 2 neutrons in the state $(0f_{5/2})$. The choice of $j_p = \frac{3}{2}$, $j_n = \frac{5}{2}$ corresponds to the lowest possible values of angular momenta. Furthermore, a relatively considerable overlapping of harmonic oscillator configurations $(1p)$ and $(0f)$ can be expected. Finally, it is believed that there are actually some nuclei with external protons and neutrons in the two configurations, respectively. This occurs in the mass region of about $A \sim 66$.

Zero-range residual interactions are assumed in the first stage. It is believed that this will be a good approximation. The finite range interactions, however, do not lead to any more computational difficulties. For the δ-forces it is sufficient to take for the interactions only a mixture of the Wigner and the Bartlett forces. The ratio of the two above named contributions is treated as a variable parameter.

Details of calculation

The wave function of a nuclear state is:

$$\Psi_J^M = \sum_{J_p J_n} A_{J_p J_n} \cdot \Psi\left((j_p)^2 J_p, (j_n)^2 J_n : JM\right). \tag{1}$$

*) Further referred to as I.

It is a particular case of the formula I (1) for two protons and two neutrons only. In this case the resultant angular momenta J_p and J_n of protons and neutrons, respectively, form a complete description of the state, and the numbers β_P, β_N are unnecessary.

Therefore, the function $\Psi((j_p)^2 J_p (j_n)^2 J_n : JM)$ can be written in the form of

$$\Psi((j_p)^2 J_p (j_n)^2 J_n : JM) = \sum_{M_p M_n} C^{JM}_{J_p M_p J_n M_n} \Psi((j_p)^2 J_p M_p) \cdot (\Psi(j_n)^2 J_n M_n), \tag{2}$$

where $C^{JM}_{J_p M_p J_n M_n}$ denotes a vector addition coefficient and $\Psi((j_p)^2 J_p M_p)$, $\Psi((j_n)^2 J_n M_n)$ are the wave functions for protons and neutrons, respectively. We have then

$$\Psi((j_p)^2 J_p M_p) = \sum_{m_\mu} C^{J_p M_p}_{j_p m j_p \mu} \cdot \Psi(j_p m) \cdot \Psi(j_p \mu), \tag{3}$$

and

$$\Psi((j_n)^2 J_n M_n) = \sum_{m_\mu} C^{J_n M_n}_{j_n m j_n \mu} \cdot \Psi(j_n m) \cdot \Psi(j_n \mu). \tag{4}$$

On account of the Pauli principle only even values of angular momenta J_p, J_n are allowed. The total angular momentum J has to fulfill the inequality: $|J_p - J_n| \leqslant J \leqslant (J_p + J_n)$. All possible configurations (J, J_p, J_n) are given in Table I. It may be seen from this table that we have 14 in-

TABLE I

Classification of allowed states

J_p	J_n	J
0	0	0
0	2	2
0	4	4
2	0	2
2	2	0, 1, 2, 3, 4
2	4	2, 3, 4, 5, 6

dependent states and the corresponding secular equation will then be of the 14-th degree. However, owing to the scalar character of the residual interactions operator, only the diagonal matrix elements in respect to J can appear. The secular equation is then reduced to a set of secular equations of a considerably lower degree, as shown in Table II.

The residual interaction operator has the form

$$V = V_{pp} + V_{nn} + \sum V_{np}, \tag{5}$$

(index p denotes proton, index n — neutron)

where

$$V_{ij} = -V_0(1-\beta+\beta\vec{\sigma}_i\cdot\vec{\sigma}_j)\,\delta(\vec{r}_i-\vec{r}_j)\,. \tag{6}$$

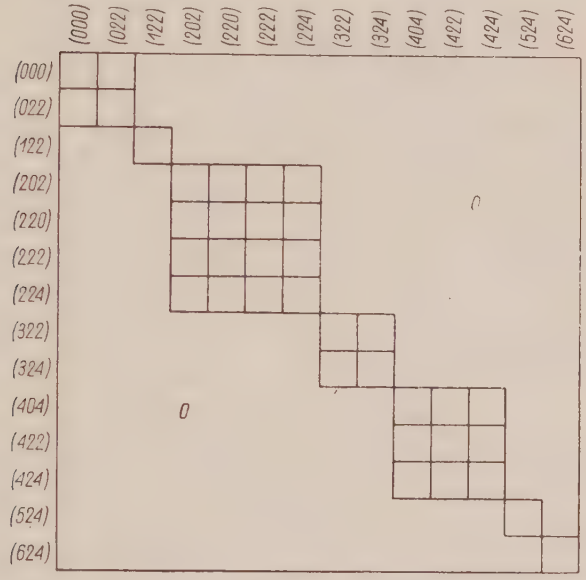
The calculation of the matrix elements of the operators V_{pp} and V_{nn} is simple:

$$\langle (j_p)^2 J'_p,\,(j_n)^2 J'_n : JM|V_{pp}|(j_p)^2 J_p,\,(j_n)^2 J_n : JM\rangle = \\ = \langle (j_p)^2 J_p|V_{pp}|(j_p)^2 J_p\rangle \cdot \delta_{J_p J'_p}\cdot \delta_{J_n J'_n} \tag{7}$$

$$\langle (j_p)^2 J'_p,\,(j_n)^2 J_n : JM|V_{nn}|(j_p)^2 J_p,\,(j_n)^2 J_n : JM\rangle = \\ = \langle (j_n)^2 J_n|V_{nn}|(j_n)^2 J_n\rangle \cdot \delta_{J'_p J_p}\cdot \delta_{J'_n J_n}\,. \tag{8}$$

TABLE II

Energy matrix in the (J, J_p, J_n)-representation



Only diagonal elements are present in the representation (J, J_p, J_n). The expressions $\langle (j_p)^2 J_p|V_{pp}|(j_p)^2 J_p\rangle$ and $\langle (j_n)^2 J_n|V_{nn}|(j_n)^2 J_n\rangle$ are calculated for the forces of the type (6) in a standard way by transformation to the LS-coupling. For the pure Wigner forces, the matrix elements (7) and (8) are given in [2].

Matrix elements of the operator $\sum V_{np}$ can be written as

$$\langle (j_p)^2 J'_p (j_n)^2 J'_n : JM|V_{np}|(j_p)^2 J_p (j_n)^2 J_n : JM\rangle = \tag{9}$$

$$= 4\cdot\{(2J'_p+1)(2J'_n+1)(2J_p+1)(2J_n+1)\}^{1/2}\cdot \sum_{J_a J_b}(2J_a+1)(2J_b+1)\cdot$$

$$\cdot \begin{Bmatrix} j_p & j_n & J_a \\ j_p & j_n & J_b \\ J'_p & J'_n & J \end{Bmatrix}\cdot \begin{Bmatrix} j_p & j_n & J_a \\ j_p & j_n & J_b \\ J_p & J_n & J \end{Bmatrix}\cdot \langle (j_p j_n) J_a|V_{np}|(j_p j_n) J_a\rangle$$

$\begin{Bmatrix} a & b & e \\ c & d & f \\ g & h & k \end{Bmatrix}$ — denotes the Wigner $(9j)$-symbols and the expressions

$$B(J'_p J'_n, J_p J_n, J_a J) = \{(2J'_p+1)(2J'_n+1)(2J_p+1)(2J_n+1)\}^{1/2} \cdot \sum_{J_b} (2J_a+1)(2J_b+1) \begin{Bmatrix} j_p & j_n & J_a \\ j_p & j_n & J_b \\ J'_p & J'_n & J \end{Bmatrix} \begin{Bmatrix} j_p & j_n & J_a \\ j_p & j_n & J_b \\ J_p & J_n & J \end{Bmatrix} \tag{10}$$

can be computed by means of numerical tables of the Wigner $(9j)$-symbols [3].

Matrix elements $\langle (j_p j_n) J_a | V_{np} | (j_p j_n) J_a \rangle$ are calculated in the usual way by transformation to the LS-coupling. Then, for the orbital parts, we obtain

$$\begin{aligned} \langle (pf)(L=2)|V_{np}|(pf)(L=2)\rangle &= F^{(0)} + \tfrac{4}{25} F^{(2)} \\ \langle (pf)(L=3)|V_{np}|(pf)(L=3)\rangle &= F^{(0)} - \tfrac{1}{5} F^{(2)} \\ \langle (pf)(L=4)|V_{np}|(pf)(L=4)\rangle &= F^{(0)} + \tfrac{1}{175} F^{(2)} . \end{aligned} \tag{11}$$

The Slater integrals $F^{(0)}(1p0f)$, $F^{(2)}(1p0f)$ can be calculated by Thieberger's method [4]. We then have

$$F^{(k)}(1p0f) = \tfrac{5}{2} F^{(k)}(0p0f) - 5F^{(k)}(0d0f) + \tfrac{7}{2} F^{(k)}(0f0f) . \tag{12}$$

The Slater integrals of the right side of (12) are given in [4].

When the matrix elements of V_{pp}, V_{nn} and V_{np} are known, the numerical diagonalization of the whole matrix is performed and the resulting excited levels are found.

Numerical results and discussion

For zero range forces the Slater integrals fulfill the relationship

$$F^{(k)} = (2k+1) F^{(0)} \tag{13}$$

for a given configuration. In order to compare the integrals for various configurations it is convenient to write them down in the asymptotic form (for zero range forces):

$$\begin{aligned} F^{(0)}(1p1p) &= \tfrac{3\,6\,3}{1\,2\,8\,0} I_0 \approx 0.28359\, I_0 \\ F^{(0)}(0f0f) &= \tfrac{4\,2\,9}{2\,2\,4\,0} I_0 \approx 0.19152\, I_0 \\ F^{(0)}(1p0f) &= \tfrac{6\,9}{6\,4\,0} I_0 \approx 0.10781\, I_0 , \end{aligned}$$

where I_0 is a quantity independent of the particular configuration (it is defined in [4]). The parameter β in (6) is treated as a variable. For the nuclei of the region considered it can be assumed (see [5]) to lie in the range

$$0.10 \leqslant \beta \leqslant 0.25 . \tag{14}$$

For the numerical computations the following three values are chosen: (i) $\beta = 0$ (i. e. the pure Wigner forces), (ii) $\beta = 0.08333$ (which corresponds to the ratio $\frac{3}{2}$ for the triplet to singulet forces) and (iii) $\beta = 0.2$. According to estimate (14) the last case is the most probable one.

In Fig. 1a level scheme for $(1p_{3/2})^2(0f_{5/2})^2$ configuration for $\beta = 0.08333$ is presented (in the arbitrary units). On the left of the picture there is a level scheme computed without configuration interaction ($V_{np} = 0$). It is a simple sum of two-level schemes found for proton and neutron pair separately. As may be seen, the configuration interaction: (i) re-

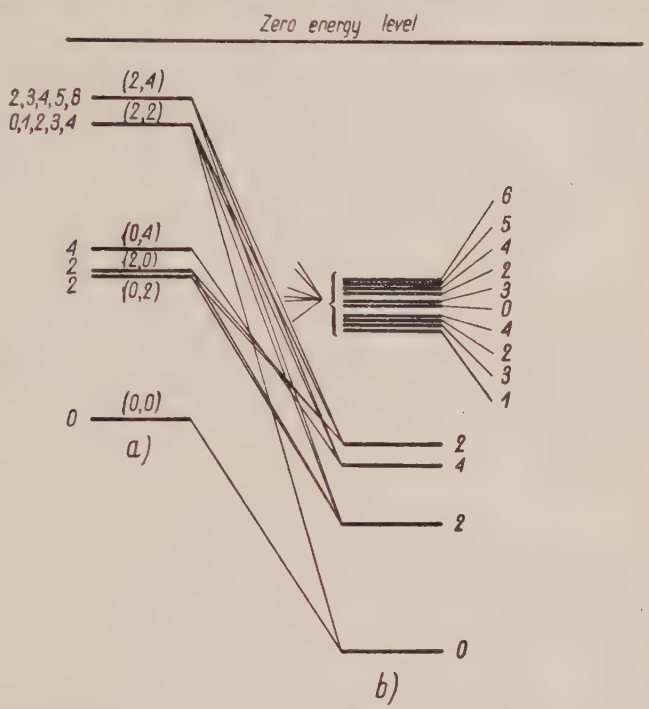


Fig. 1. Level schemes for the $(1p_{3/2})^2$ $(0f_{5/2})^2$ configuration (= 0.0–+++); a) without configurational interaction; b) with configurational interaction

moves the degeneracy with respect to the total angular momentum J ($|J_p - J_n| \leqslant J \leqslant J_p + J_n$), (ii) leads to the configurational mixing of various states (J_p, J_n), in particular to the repulsion of levels with the same total angular momentum J, (iii) shifts the whole scheme downwards. The distance between the "zero-energy" line and a ground state level corresponds to the binding energy of the group of 4 external nucleons (i. e. the α-energy). In the case of $V_{np} = 0$ it would merely be a sum of pairing energies of protons and neutrons.

The above considerations are applied with some approximation to the nucleus $_{30}Zn^{66}$. From the experimental data [6]-[8] for ground state angular momenta of the neighbouring nuclei the following configuration for Zn^{66} may be expected: double closed shell core $(28+28)$ + protons $(1p_{3/2})^2$ + neutrons $(1p_{3/2})^4(0f_{5/2})^4$. On the assumption that both neutron subshells $(1p_{3/2})^4(0f_{5/2})^6$ are put into the core we obtain the configurations $(1p_{3/2})^2$, $(0f_{5/2})^{-2}$ for external protons and neutrons respectively.

In this case nuclear level schemes for various β are given in Fig. 2. To compare them the strength of the residual interaction potential is chosen in such a way that the energies of the first excited level (with spin 2) are equal in all three cases.

It is found by numerical computations that, in the case of the pure Wigner forces ($\beta = 0$), when nondiagonal elements of the operator V are small in comparison with the diagonal ones, the effect of the mixed configurations is rather weak. Hence the classification of excited levels according to the seniority quantum number is roughly conserved. It can easily be seen from Fig. 2 that the whole scheme in this case may

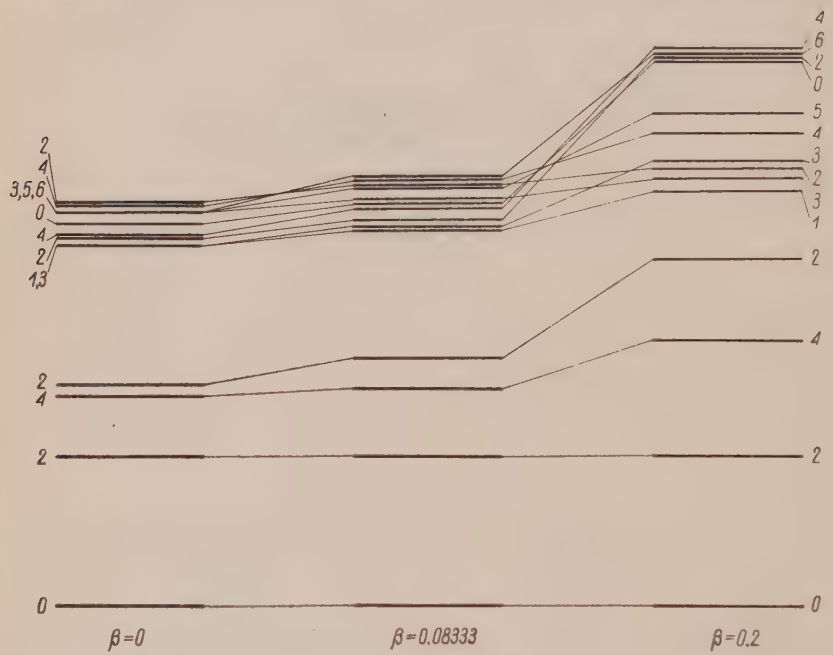


Fig. 2. Level schemes for the $(1p_{3/2})^2$ $(0f_{5/2})^2$ configuration

be divided into three groups: 1) the ground state level (seniority 0), 2) the next three excited levels (seniority 2), and 3) the remaining levels (seniority 4). In the other case of the pure triplet forces ($\beta = 0.25$) the matrix elements of V_{pp} and V_{nn} vanish, because of zero range forces. Then the matrix elements of the V_{np}-operator are most important. The configurational interaction plays an essential part in this case. Taking into account inequality (12) such a situation can be expected to be a very good approximation to reality.

In the above considerations, the residual interaction operator was treated as a small perturbation of one-particle states. However, in the neighbourhood of the Zn^{66} nucleus there are three one-particle levels lying close: $(1p_{3/2})$, $(0f_{5/2})$, and $(1p_{1/2})$. The mutual interference of these

three states is then possible. This has not, however, been taken into account, and undoubtedly constitutes the greatest disadvantage of the considerations presented above. Therefore one can expect that the above method will lead to correct results only for the first few low lying levels. For higher excitations the $j_p = \frac{3}{2}$ and $j_n = \frac{5}{2}$ are no longer good quantum numbers. On the other hand, however, it is well known that the nucleon pairs are not "inclined" to jump up on the next excited one-particle level as almost all deformed even-even nuclei are known to have no low lying two-particle levels, though those of the odd-A type – do have them. This may serve as an argument for employing the above method in the work.

In Fig. 3, a comparison of the theoretical results (for $\beta = 0.2$) with the experimental data is given for the first few levels (there is no agreement for higher levels). The theoretical scheme is fitted to the experimental one in such a way as to get equal energies of the first excited level (with spin 2).

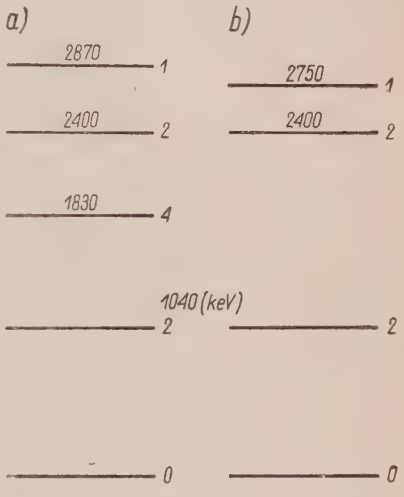


Fig. 3. Level schemes in Zn^{66} a) calculated; b) experimental

It may be seen that the level with spin 4 does not occur in the experimental scheme. The experimental data, however, were obtained from the decay schemes of the neighbouring nuclei with low ground state angular momenta. Thus it might be that the experimental detection of such high spin level would be difficult because of the relatively considerable degree of forbidenness.

The author is sincerely indebted to Professor L. A. Sliv for continued encouragement in the course of this work.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

Transition Matrices for Reactions between Two Bosons and Two Fermions

by

J. WERLE

Presented by L. INFELD on May 7, 1958

A general expression for the transition matrix M describing arbitrary processes between two bosons and two fermions has been given in a recent paper [1]. It has the form

$$M = \int \delta_{(4)}(p_1 + p_2 - p_3 + p_4) \prod_{A=1}^{4} \{\delta(p_A^2 - m_A^2)\, dp_A\} \cdot$$

$$\cdot \{\Phi_{1r}\Phi_{2s}\overline{\Psi}_{3u}\Gamma^{rsuv}\Psi_{4v} + \overline{\Psi}_{4v}\hat{\Gamma}^{rsuv}\Psi_{3u}\Phi_{2s}\Phi_{1r}^{+}\,, \tag{1}$$

where the Γ's have the following structure

$$\Gamma^{rsuv} = \sum_{K=1}^{2N} (a_K + \gamma_5 a'_K)\, G_K^{rsuv},$$

$$\hat{\Gamma}^{rsuv} = \sum_{K=1}^{2N} \gamma_4 G^{rsuv} + \gamma_4(b_K - \gamma_5 b'_K)\,. \tag{2}$$

Notations used here are the same as in [1]. The coefficients a_K, a'_K, b_K, b'_K are scalar functions of the four-momenta p_K^μ. Linearly independent covariant forms G_K^{rsuv} are constructed from the p_A^μ, and $g^{\mu\nu}$ only; they do not contain the pseudotensor $\varepsilon^{\alpha\beta\gamma\delta}$ or any complex numbers. Besides tensor indices $(rsuv)$ the G's contain of course two spinor indices, which will not be denoted explicitly. Now we shall investigate the form of the G's in greater detail.

First we observe that, if G_1^{rsuv} is a covariant which satisfies all the requirements listed above, then $G_2^{rsuv} = (\gamma p_1) G_1^{rsuv}$ is another good covariant independent of G_1^{rsuv}. Thus, we see that the set of $2N$ linearly independent covariants G_K^{rsuv} can be split into two groups differing only

by the factor γp_1. It is to be noted that there are no other non-trivial factors, since, because of the Dirac equations

$$(3) \qquad (\gamma p_3 + m_3)\Psi_3^u = 0 \qquad (\gamma p_4 + m_4)\Psi_4^v = 0$$

and energy-momentum conservation, the factors $\gamma p_2, \gamma p_3, \gamma p_4$, etc. can be reduced either to some constants which can be absorbed into the scalar coefficients, or to γp_1. Therefore, we can write Γ^{rsuv} and $\hat{\Gamma}^{rsuv}$ in the following form [2].

$$(4) \qquad \Gamma^{rsuv} = \sum_{K=1}^{N} \{a_K + \gamma_5 a'_K + (\bar{a}'_K + \gamma_5 \bar{a}'_K)\gamma p_1\} G_K^{rsuv}$$

$$(5) \qquad \hat{\Gamma}^{rsuv} = \sum_{K=1}^{N} \gamma_4 G_K^{rsuv+} \gamma_4 \{b_K - \gamma_5 b'_K + \gamma p_1 (\bar{b}_K - \gamma_5 \bar{b}'_K)\},$$

where the G's occurring in (4) and (5) do not have any more common factors. In this way the number of different independent covariants to be found for any given type of reaction is reduced by a factor two.

Let us now discuss some examples:

1) For reactions of the type $(0, 0, \frac{1}{2}, \frac{1}{2})$, i. e. for reactions between two Bosons of spin 0 and two Fermions of spin $\frac{1}{2}$ there is only one term in the sum (4) or (5). Thus, we have only covariant G which is a scalar and can be taken to be equal to unity:

$$(6) \qquad N = 1, \qquad G = 1.$$

The general forms of Γ and $\hat{\Gamma}$ contain, therefore, only two parity conserving and two parity non-conserving scalar coefficients.

2) For reactions of the type $(0, 1, \frac{1}{2}, \frac{1}{2})$, i. e. for reactions between one Boson of spin 0, one Boson of spin 1 and non-vanishing mass, and two Fermions of spin $\frac{1}{2}$ we have three independent covariants

$$(7) \qquad N = 3, \qquad G_1^\mu = p_1^\mu, \qquad G_2^\mu = p_3^\mu, \qquad G_3 = \gamma^\mu.$$

The general forms of Γ and $\hat{\Gamma}$ contain, therefore, six parity conserving and six parity non-conserving scalar coefficients.

3) If, in the reaction $(0, 1, \frac{1}{2}, \frac{1}{2})$, the spin 1 particle is a photon, we have some additional restrictions due to the gauge invariance of the theory. Thus, we find

$$(8) \qquad N = 2, \qquad G_1^\mu = p_1^\mu (p_2 p_3) - p_3^\mu (p_1 p_2), \qquad G_2^\mu = \gamma^\mu (p_1 p_2) - p_1^\mu (\gamma p_2).$$

The general forms of Γ and $\hat{\Gamma}$ contain in this case four parity conserving and four parity non-conserving scalar coefficients.

4) For reactions of the type $(1, 1, \frac{1}{2}, \frac{1}{2})$, i. e. for reactions between two Bosons of spin 1 and non-vanishing masses and two Fermions of spin $\frac{1}{2}$ we have ten independent covariants

$$N = 10\,, \tag{9}$$

$$G_1^{\mu\nu} = g^{\mu\nu}, \quad G_2^{\mu\nu} = p_2^\mu p_1^\nu, \quad G_3^{\mu\nu} = p_2^\mu p_3^\nu, \quad G_4^{\mu\nu} = p_3^\mu p_1^\nu, \quad G_5^{\mu\nu} = p_3^\mu p_3^\nu,$$

$$G_6^{\mu\nu} = \gamma^\mu p_1^\nu, \quad G_7^{\mu\nu} = \gamma^\mu p_3^\nu, \quad G_8^{\mu\nu} = p_2^\mu \gamma^\nu, \quad G_9^{\mu\nu} = p_3^\mu \gamma^\nu, \quad G_{10}^{\mu\nu} = \gamma^\mu \gamma^\nu .$$

The Γ's contain, therefore, twenty parity conserving and twenty parity non-conserving scalar coefficients.

5) If, in the reaction $(1, 1, \frac{1}{2}, \frac{1}{2})$, one of the Bosons is a photon, the number of independent covariants is reduced by gauge invariance requirement to seven. Thus,

$$N = 7\,, \tag{10}$$

$$G_1^{\mu\nu} = g^{\mu\nu}(p_1p_2) - p_2^\mu p_1^\nu, \quad G_2^{\mu\nu} = \{p_2^\mu(p_1p_3) - p_3^\mu(p_1p_2)\}p_1^\nu,$$

$$G_3^{\mu\nu} = \{p_2^\mu(p_1p_3) - p_3^\mu(p_1p_2)\}p_3^\nu, \quad G_4^{\mu\nu} = \{\gamma^\mu(p_1p_2) - p_2^\mu(\gamma p_1)\}p_1^\nu,$$

$$G_5^{\mu\nu} = \{\gamma^\mu(p_1p_2) - p_2^\mu(\gamma p_1)\}p_3^\nu, \quad G_6^{\mu\nu} = \{p_2^\mu(p_1p_3) - p_3^\mu(p_1p_2)\}\gamma^\nu,$$

$$G_7^{\mu\nu} = \{\gamma^\mu(p_1p_2) - p_2^\mu(\gamma p_1)\}\gamma^\nu .$$

It can easily be seen that all these G's satisfy gauge invariance condition

$$p_1 G_K^{\mu\nu} = 0\,. \tag{11}$$

These are just a few examples which show the actual forms of the covariants G_K^{rsuv} in the case of spin $0, \frac{1}{2}$ and 1 particles. Knowing the set of N basic independent covariants G_K^{rsuv} we can write the most general form of Γ^{rsuv} and $\hat{\Gamma}^{rsuv}$ as a linear superposition of form (2) with scalar expansion coefficients.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 7 1958

THEORETICAL PHYSICS

On some Application of the OPW Calculations

by

M. SUFFCZYŃSKI

Presented by L. INFELD on May 30, 1958

The results of calculations of band structure by the orthogonalized plane wave (OPW) method have been helpful in correlating results of electrical, magnetic and optical measurements on the crystalline solids. Usually only the band structure, i. e. the electron energy eigenvalues are determined by the OPW method [1]–[3] but it is possible also to go one step further and to calculate the corresponding crystal wave functions as the eigenvectors to these eigenvalues. This step is necessary, e. g. when the task is undertaken to compare the final electronic charge distribution with the initial one used in constructing the crystal potential [4].

The eigenvalues of energy obtained by the OPW method are considered often as quite reliable, especially if sufficient convergence is achieved. We may hope that the corresponding wave functions will be reliable, too, and will describe the states of electrons in the valence and the lowest conduction bands satisfactorily. The question arises whether these wave functions could be used in practice for calculating from the first principles the matrix elements of various one-electron operators, relevant in theoretical estimate of the absolute magnitude of certain physical quantities. The most important in optics is apparently the momentum matrix element. We try to see what amount of labour would be involved in calculating it with OPW functions.

We consider the one-electron states, described by wave functions Ψ and Ψ' which are eigenfunctions of the crystal Hamiltonian, belonging to different eigenvalues but to the same wave vector $\boldsymbol{k}$.

We assume that both functions can be adequately represented by linear combination of orthogonalized plane waves with coefficients $a_m(\boldsymbol{k})$:

$$\Psi = \sum_m a_m(\boldsymbol{k})\, \chi_m(\boldsymbol{k}, \boldsymbol{r}),$$

$$\Psi' = \sum_{m'} a'_{m'}(\boldsymbol{k})\,\chi_{m'}(\boldsymbol{k}, \boldsymbol{r})\,.$$

An orthogonalized plane wave is

$$\chi_m(\boldsymbol{k}, \boldsymbol{r}) = (N\Omega)^{-\frac{1}{2}}\Big[e^{i(\boldsymbol{k}+\boldsymbol{k}_m)\boldsymbol{r}} - \sum_l b_l(\boldsymbol{k}+\boldsymbol{k}_m) \sum_\nu e^{i(\boldsymbol{k}+\boldsymbol{k}_m)\boldsymbol{r}_\nu}\varphi_l(\boldsymbol{r}-\boldsymbol{r}_\nu)\Big]\,.$$

We assume that in Ψ and Ψ' the plane waves are orthogonalized to the same set of core functions; $\boldsymbol{k}$ is a general point in momentum space, $\boldsymbol{k}_m$ is the reciprocal lattice vector, $\boldsymbol{r}$ is a general point in configuration space, $\boldsymbol{r}_\nu = \boldsymbol{r}_n + \boldsymbol{t}_j$ is the vector to the j-th atom in the n-th cell. The integers $n_1, n_2, n_3 = \boldsymbol{n}$ run over all N cells in the crystal. The index $j = 1, 2, \ldots s$ labels the s atoms in the unit cell. The volume of the unit cell is Ω.

The core atomic functions are $\varphi_l(\boldsymbol{r})$ with the set of quantum numbers denoted by one letter l. Orthogonality coefficients $b_l(\boldsymbol{k})$ are given by

$$b_l(\boldsymbol{k}) = \int_\infty e^{i\boldsymbol{k}\boldsymbol{r}}\varphi_l^*(\boldsymbol{r})\,d\boldsymbol{r}\,.$$

We consider the matrix element between the states Ψ and Ψ' of the momentum operator projected on polarization vector $\boldsymbol{e}$. We shall assume for simplicity that the core functions on different atom sites do not overlap, that is,

$$\int_\infty \varphi_l^*(\boldsymbol{r}-\boldsymbol{r}_\nu)\varphi_{l'}(\boldsymbol{r}-\boldsymbol{r}_{\nu'})\,d\boldsymbol{r} = 0, \qquad \text{for} \qquad \nu \neq \nu'\,,$$

and moreover

$$\int_\infty \varphi_l^*(\boldsymbol{r}-\boldsymbol{r}_\nu)\boldsymbol{\nabla}\varphi_{l'}(\boldsymbol{r}-\boldsymbol{r}_\nu)\,d\boldsymbol{r} = 0 \qquad \text{for} \qquad \nu \neq \nu'\,.$$

Neglect of overlapping of the core functions is often done in OPW calculations [2], [5]. A refinement possible, e. g. along the lines of the tight-binding approximation would necessitate more elaborate formulae. With this only neglect we can write for the momentum matrix element

$$-i\int_\infty \Psi^*\boldsymbol{e}\boldsymbol{\nabla}\Psi'\,d\boldsymbol{r} = \frac{-1}{\Omega}\sum_m\sum_{m'} a_m^*(\boldsymbol{k})a'_{m'}(\boldsymbol{k})\times$$

$$\times\Big[\boldsymbol{e}(\boldsymbol{k}_m+\boldsymbol{k}_{m'}+2\boldsymbol{k})\sum_l b_l^*(\boldsymbol{k}+\boldsymbol{k}_m)b_l(\boldsymbol{k}+\boldsymbol{k}_{m'}) +$$

$$+\,i\sum_l{}'\sum_{l'} b_l^*(\boldsymbol{k}+\boldsymbol{k}_m)b_{l'}(\boldsymbol{k}+\boldsymbol{k}_{m'})\int_\infty \varphi_l^*(\boldsymbol{r})\boldsymbol{e}\boldsymbol{\nabla}\varphi_{l'}(\boldsymbol{r})\,d\boldsymbol{r}\Big]\times\sum_{j=1}^{s} e^{i(\boldsymbol{k}_{m'}-\boldsymbol{k}_m)\boldsymbol{t}_j}\,.$$

Since the number of functions to which orthogonalization is required is always finite the summations inside the brackets are rigorously finite.

For example, in Si there are four and in Ge fourteen core functions. But also the summations over $\boldsymbol{m}$ and $\boldsymbol{m}'$ are, in practice, finite, because usually a finite number of orthogonalized plane waves renders convergent energy eigenvalue. The number of $a_m(\boldsymbol{k})$ terms is dictated first by symmetry of the $\boldsymbol{k}$-point and secondly by requirements of convergence.

If, following Slater's [6] and Löwdin's [7] procedure, the core atomic functions φ_l are approximated by analytical expressions, the integrals $\int_\infty \varphi_l^* \mathbf{r} \varphi_{l'} \, d\boldsymbol{r}$ can be calculated, in principle, easily, and moreover the orthogonality coefficients $b_l(\boldsymbol{k})$ are given as analytical functions of $\boldsymbol{k}$. This has some advantage insofar as, in the applications, the square of the absolute value of momentum matrix element must be summed over $\boldsymbol{k}$ with appropriate energy denominator or delta function. That is, an integration over a three- or two-dimensional region of $\boldsymbol{k}$-space must be performed, and moreover a summation over the different excited states. These summations make the whole problem prohibitively difficult: in fact, the OPW procedure can be accomplished in the $\boldsymbol{k}$-points of high symmetry only, and the energy eigenvalues and the coefficients $a_m(\boldsymbol{k})$ are obtainable in purely numerical form.

However, in such problems as in the calculation of the absorption edge in semiconductors, the $\boldsymbol{k}$-integration extends practically over a small $\boldsymbol{k}$-range [8]. The band extrema lie sometimes at the $\boldsymbol{k}$-points of high symmetry. In Si the minimum in the first conduction band is near (100) point, [9], in Ge the minima in the lowest conduction band are along (111) axes and at (000) point, [3], [9], in In Sb the valence band maximum and the conduction band minimum both are at (000) point, [10]. It appears therefore reasonable to try to calculate in such instances the momentum matrix element at these particular $\boldsymbol{k}$-points just to see once its magnitude, and not to take it simply as an undetermined constant.

The formula for the matrix element obtained by use of the OPW method displays clearly the fact pointed by Herring [11], [12] that the transitions forbidden, e. g. in the simple tightbinding approximation, may be different from zero in a more realistic approximation. The properties of transition probabilities due to band symmetries [13] can be investigated with the help of the explicit expression for the matrix element in terms of OPW's.

In calculating the optical properties of metals the inadequacy of using a constant matrix element has been already remarked by Sergeiev and Tchernikovsky [14]. They have applied to that problem the approximation of nearly free electrons. In this approximation the absolute value of the momentum matrix element in each $\boldsymbol{k}$-point is controlled by the absolute magnitude of the Fourier coefficient of the crystal potential with particular indices. In monovalent alkali and noble metals

the momentum matrix element between the nearly free electron waves varies considerably throughout the occupied region in $\boldsymbol{k}$-space.

The examples calculated by Wilson [15] show that it varies by a factor of 20 in Ag. The virtue of the nearly free electron approximation lies in its simplicity, which allows, after some neglect, all the necessary summations over $\boldsymbol{k}$ to be performed analytically. It is, in fact, this approximation which has been used in the calculations of optical constants of metals in the ultraviolet region [16] and also in the related problems of electron losses in metals [17].

Perhaps the modified plane wave method proposed recently [18] might be applicable for calculating matrix elements since it is similar to the OPW method, and comparatively simple.

I thank Professor H. Jones for the kind hospitality extended to me at the Imperial College.

I thank Dr. E. P. Wohlfarth for a discussion.

The present paper was written in London while on leave of absence from the Institute of Theoretical Physics, Warsaw, Poland.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 7

В настоящей работе приводится очерк некоторой теории G, аксиомы которой зависят от числовых параметров. Придавая этим параметрам, соответствующим образом подобранные значения, можем получить модели разных известных геометрий, как напр., модель афинной n-мерной геометрии, проективной n-мерной геометрии, n-мерной геометрии Мэбкса или Лиевой n-мерной геометрии. К сожалению, для данных параметров система аксиом теории G не является полной системой аксиом соответствующей геометрии и нуждается в пополнении некоторыми добавочными аксиомами, характеризующими эту геометрию.

В работе ставится несколько новых проблем.

Ожидается, что дальнейшее изучение теории G может не только обнаружить некоторые связи между различными геометриями, или дать возможность доказывать общие теоремы для разных геометрий, но также (при соответствующем подборе параметров) получить некоторые новые геометрии со свойствами, аналогичными свойствам уже известных геометрий.

В работе вводится понятие E-компактного пространства: пространство X называется E-компактным, если не существует пространство Y, которое содержит X в качестве собственного плотного множества и которое обладает свойством таким, что всякая функция $f \in E^X$ допускает расширение $f^* \in E^Y$.

Это понятие является обобщением понятия компактных пространств (при E=единичный замкнутый интервал) и Q-пространств в смысле Гюитта [1] (при E = числовая прямая).

Доказаны следующие теоремы:

1. *Замкнутое подмножество E-компактного пространства тоже E-компактно.*

2. *Топологическое произведение E-компактных пространств тоже E-компактно.*

3. *Пространство X E-компактно тогда и только тогда, когда оно гомеоморфно замкнутому подмножеству некоторой топологической степени пространства E.*

В работе даны обобщения результатов Колмогорова [8], Смирнова [12], Долскера [3], Фиша [6], а также Чжана и Фиша [2]. Приводится здесь обобщение теоремы Колмогорова.

Теорема 2. Пусть $\{Y_k(t), 0 \leqslant t \leqslant 1\}$ $(k = 1, 2, \ldots)$ *будет последовательностью действительных, сепарабельных, независимых и одинаково распределенных стохастических процессов и пусть выполняются соотношения* (1)-(3), *причем* $a(t) = t$ *и* $b(t) = 1 - t$. *Тогда для любого* $\lambda > 0$

$$\lim_{n\to\infty} P^{\xi_n}(\sup_t |\xi_n(t)| < \lambda) = \sum_{r=-\infty}^{\infty} (-1)^r \exp(-2\lambda^2 r^2),$$

где

$$\xi_n(t) = \sqrt{n} \cdot \frac{1}{n} \sum_{k=1}^{n} \Big[Y_k(t) - E Y_k(t) \Big],$$

а P^{ξ_n} *является вероятностной мерой, индуцированной конечномерными распределениями процесса* $\xi_n(t)$ *в пространстве* $D[0, 1]$ *Скорохода-Прохорова.*

Заметка содержит следующий результат: в множестве с мощностью $\leqslant 2^{\aleph_0}$ каждая метрика является по существу максимальным σ-идеалом в соответствующим способом определенной σ-алгебре подмножеств множества всех сепарабельных метрик и, наоборот, каждый такой σ-идеал определяет метрику. Взаимно однозначное соответствие максимальных σ-идеалов и метрик получено методами теории вероятностных процессов и выражается математическими ожиданиями.

В первой части работы (пкт. 2 и 3) рассматриваются общие алгебры с бесконечными действиями. Приводятся в частности: форма субалгебры, порожденной данным множеством, теоремы о мощности субалгебр, а также определение и свойства алгебры абсолютно свободной.

В другой части работы (пкт. 4, 5 и 6), используя понятие алгебры абсолютно свободной, дается алгебраическая конструкция логики $P_{\eta,\tau}$, соответ-

ствующей классу одноименных моделей с бесконечными действиями и отношениями.

В логике $P_{\eta,\tau}$ имеются конечные и бесконечные альтернативы, а также конъюнкции длиной меньше, чем $\aleph_\eta$; автор пользуется $\aleph_\tau$ переменными, но не прибегает к квантификаторам. Теоремы (6.1) и (6.2) дают характеристику классов моделей определяемых в логике $P_{\eta,\tau}$.

В пкт. 5 работы дана дедукция, произведенная на подмножествах множества $O \subset P_{\eta,\tau}$, состоящего из всех выражений, содержащих лишь импликации и отрицания. Рассматривается следствие O, опирающееся на аксиоматическую директиву исчисления предложений, директиву отрывания и подстановки, а также следствие $O=$, опирающееся, кроме того, на инфинитистическую директиву тождества.

Для систем O доказана теорема Гёделя, согласно которой всякая не-противоречивая система O имеет модель. В работе приводится пример не-противоречивой системы $O=$, которая не имеет модели с тождеством, а также условия необходимые и достаточные для того, чтобы упомянутая система имела модель с тождеством.

3. СЕМАДЕНИ, О ЛИНЕЙНЫХ ФУНКЦИОНАЛАХ В KB-ЛИНЕАЛАХ И О ПРЕДСТАВЛЕНИИ ФУНКЦИОНАЛОВ В ВИДЕ ИНТЕГРАЛОВ стр. 457—462

Автор рассматривает KB-линеалы, в которых кроме нормы $\| \ \|$ определено некоторое понятие сходимости $x_n \xrightarrow[l]{} x_0$ типа $\mathcal{L}$ Фреше, причем предполагается, что $\|x_n\| \to 0$ влечет за собой $x_n \xrightarrow[l]{} 0$, а $x_n \xrightarrow[l]{} 0$ влечет за собой $|x_n| \xrightarrow[l]{} 0$. Кроме того, предполагаем непрерывность линейных операций и теорему о трех последовательностях. Затем вводится сходимость, являющаяся обобщением бинормной сходимости:

$$x_n \xrightarrow[\gamma]{} x_0, \quad \text{если} \quad x_n \xrightarrow[l]{} x_0 \quad \text{и} \quad \sup_n \|x_n\| < \infty .$$

Доказывается, что множества Ξ_1 и Ξ_γ l-линейных и γ-линейных функционалов являются, соответственно, подструктурами пространства Ξ (сопряженного с $\langle X, \| \ \| \rangle$) и что множество Ξ_γ замкнуто.

В качестве приложения автор рассматривает интегральные представления γ-линейных функционалов в двух конкретных случаях: 1^0 в пространстве простых функций на булевой алгебре, где $\|x\|$ есть „*essential supremum*" функции $|x|$, а $x_n \xrightarrow[l]{} x_0$ есть сходимость почти всюду (понятия „существенная" и „почти всюду" понимаются относительно некоторого σ-идеала R), 2^0 в пространстве функций, принимающих действительные значения, непрерывных и ограниченных на некотором множестве $E = \bigcup_{n=1}^{\infty} E_n$, где E_n компактные множества, $\|x\| = \sup_E |x(t)|$, а $x_n \xrightarrow[l]{} x_0$ есть равномерная сходимость на каждом множестве E_n.

В работе авторы вывели общую ковариантную формулу на эффективное сечение для меллеровского рассеяния произвольно поляризованных электронов. Далее, авторы рассмотрели частный случай поперечной поляризации. Для этого случая найдены экспериментальные условия, при выполнении которых поляризационные эффекты проявляются максимально. В лабораторной системе максимум этих эффектов приходится в окрестности угла рассеяния $\theta = \pi/4$; равномерно с возрастанием энергии падающих электронов этот максимум перемещается в направлении меньших углов, а его значение уменьшается (рис. 2).

В работе приводится простой числовой пример применения приведенного ранее метода вычисления возбужденных уровней сферических ядер. Предполагается, что ядро содержит два внешних протона в конфигурации $(1p_{3/2})$ и два внешних нейтрона в конфигурации $(0f_{5/2})$. Произведена диагонализация полного оператора остаточных взаимодействий в представлении, для которого хорошими квантовыми числами являются результирующие моменты количества движения протонов и нейтронов. Как остаточное взаимодействие принята смесь сил Вигнера и Бартелета с нулевым радиусом действия. Эту схему уровней применено для ядра Zn^{66}. Для нескольких самых низких возбужденных уровней получено приближенное согласие с экспериментальными данными.

В работе точнее продискутирован, данный ранее, общий вид матриц перехода для реакций между двумя бозонами и двумя фермионами. Дается явный вид тензоров базиса G^{rsuv} для пяти разных типов реакций между частицами со спинами 0, $\frac{1}{2}$ и 1.

В настоящей работе дается формула матричного элемента импульса, вычисленная при применении собственных функций, полученных методом О. П. В. Матричный элемент импульса играет существенную роль в оптике кристаллов.

Поясняются приближения, которые позволяют ввести этот матричный элемент импульса в виде удобном для дальнейших расчетов и показана возможность его применения в конкретных оптических задачах, как напр., при определении положения края поглощения.

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